THE UNIVERSITY OF ALBERTA

FUNCTIONAL ANALYSIS OF A CLASS OF NONLINEAR SYSTEMS WITH RANDOM INPUTS

by

© MOHAN BIR SINGH

A THESIS

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ABSTRACT

This thesis is concerned with the analysis of a class of nonlinear systems with random inputs. The nonlinear feedback system is considered to be a perturbed version of a linear, time invariant feedback system whose open loop transfer function has poles only in the left half plane.

A functional analysis approach involving the use of random contraction mapping and the fixed point principle is considered, while restricting the input to the class of stationary, ergodic and Gaussian stochastic processes.

An analysis of a first order phase locked loop system is presented utilizing the above approach and the Volterra series expansion technique.

A comparison of the results with those obtained by other perturbation techniques indicates this approach to be a more realistic one near the threshold region. The proposed method is general in nature and can be extended to cases where a low pass filter is included in the phase lock loop.

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ERRATA

1. We want to show that the condition (given on page 28)

$$||x(t)|| = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} |x(t)|dt$$

implies $||x|| = 0$ if and only if $x = 0$.

The space of random variables with finite first order absolute moment, $E\ |\xi| < \infty$ is a Banach space with $||\xi|| = E\ |\xi|$.[22] In the case of strictly stationary ergodic processes, according to Doob[8] (p. 516), we have

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} x_t(\omega)dt = E(x_o).$$

Since $\{x_t\}$ being a strictly stationary process implies $\{|x_t|\}$ is also a strictly stationary process, we have

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} |x_t(\omega)|dt = E(|x_o|).$$

Thus $||x|| = 0$ implies $E(|x_o|) = 0$ and this can happen only if $x_o = 0$. Now utilising the property of strict stationarity,

$$E\ |x_o| = E\ |x_t| = 0 \qquad \text{for all } t.$$

Hence $||x|| = 0$ implies $x = 0$ (almost everywhere) for all t. Also if $x = 0$, then $||x(t)|| = 0$.

2. We make the assumption on pages 31 and 32 that $||xy|| \le ||x||\ ||y||$.

CHAPTER I

INTRODUCTION

The designer of control systems invariably has to deal with nonlinear phenomena. Only over a limited range can linear relations describe the real elements of such a system. The dynamic properties of control systems can be considerably improved by the introduction of nonlinear techniques. Similarly, adaptive optimal systems point up the significance of nonlinear relations. Statistical methods of design are coming increasingly into use because one can construct and study systems which successfully combat interference and which work reasonably well, not only for several common fixed signals, but for a whole spectrum of possible situations that may arise in real conditions.

In the analysis or synthesis of nonlinear systems, just as in linear systems, the first problem one encounters is how to characterize the systems. One possible way to describe nonlinear systems is by nonlinear differential equations. But the equation only gives an implicit relationship between the input and output. The other classical techniques of treating nonlinear problems are the phase plane method and the describing function method. The phase plane method can only treat second order equations effectively. The describing function method is only useful to determine the sinusoidal steady state solution for the systems in which the first harmonic is the only significant term. Recently, stability theory based on Liapunov's second method has also been applied to the analysis and synthesis of nonlinear control systems[1].

A new approach to the description of a system is through the concept of a functional series. The transfer function concept plays an important role in the analysis and synthesis of linear systems. The Volterra series is the generalization of this concept of nonlinear systems[2]. This representation leads to an explicit input-output relation for nonlinear systems described by differential equations with a forcing term. The relation consists of an infinite series composed of terms of the form of convolution integrals. The first order kernel is the impulse response of the linear portion of the nonlinear differential equation of the system. The n^{th} order term may be considered as an n-fold convolution containing the n^{th} order kernel multiplied by an n^{th} order product of the forcing term.

1-1 Statistical Models of Nonlinear Systems With Random Inputs

Consider the nonlinear automatic control system whose block diagram is shown in Fig. 1.1.

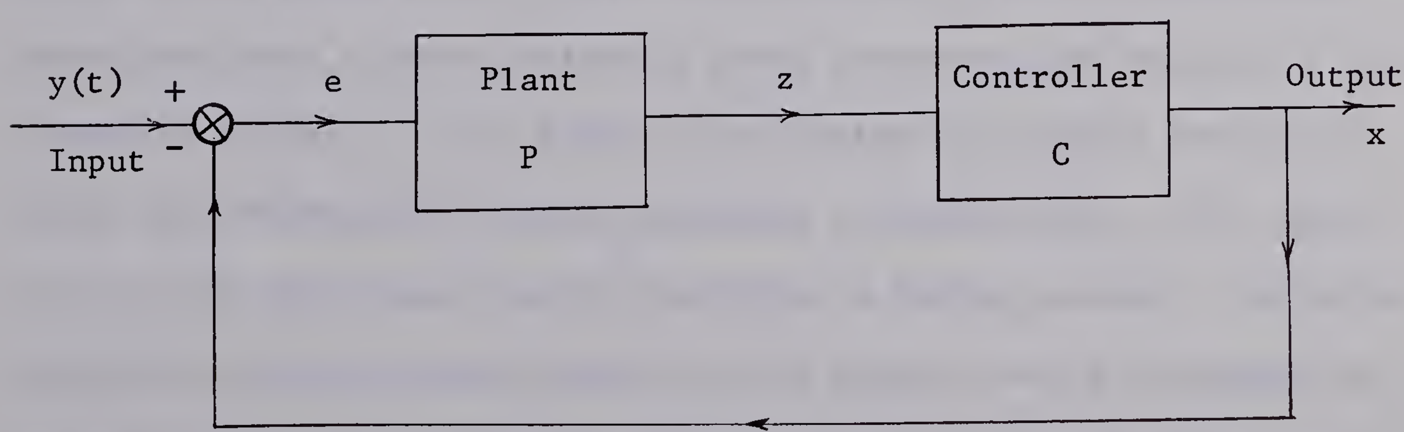

Figure 1.1

In general, both the control plant P and the controller C are described by nonlinear operators with memory. An arbitrary random signal y(t) with known statistical properties acts on the system as the input. The problem is to determine statistical characteristics of the system output x(t) with specified C and P. At present, three methods of solving the analysis problem exist[3].

(a) Functional Series Approach

The mathematical basis consists of representing the solution of the nonlinear system as a functional series. The nonlinear functional is expanded into an orthogonal series in a selected system of orthogonal polynomial functions[4]. The analysis problem is reduced to determining the parameters of the representation. This can be done by computing the cross-correlations of the outputs of the actual nonlinear operator and of the model when signals of the "white noise" type act on them. It is possible to construct an exact model of the arbitrary nonlinear operator but it is extremely laborious.

(b) Method of Constructing a Markov Model of the Nonlinear System

The response of any dynamic system can be represented by a point in the phase space. With increasing time, the representative point describes a phase trajectory which represents the history of the dynamical system. If the input to the system is a random function of time, the representative point undergoes a random motion. The trajectory in the phase space can be considered a Markov process. The probability distributions associated with the process can be determined by using Kolmogarov or Fokker-Planck equations[5]. In practice, however, numerical results can be obtained with relative ease only for the

systems described by the first or second order differential equations.

(c) Method of Statistical Linearisation

The nonlinear system is approximated by its linear statistical equivalent[6]. This method has been applied with a degree of success to open loop, zero memory nonlinear systems. Its application to closed loop system analysis is in general limited to the case in which the plant P is a linear system and controller C is a zero memory nonlinear function.

The impetus to the first method of analysis was given by the work of Wiener[4]. Its subsequent development and application have been documented in the monograph of Smith[7]. The present analysis employs a functional-theoretical approach to the problem rather than the infinite series Hermite Laguerre expansion of Wiener et al. The problem is restricted to a stationary Gaussian ergodic process as an input to the nonlinear feedback system.

The second chapter gives a review of this class of processes and indicates the use of the random fixed point theorem and the contraction mapping principle in the analysis of the problem. In the next chapter, a specific nonlinear feedback system consisting of a linear differential operator with zeros only in the left half plane and a "square" nonlinearity in the feedback loop is considered. The constraints on the input and the output are developed through the use of the contraction mapping principle. The fourth chapter deals with the analysis of phase-locked loop systems which are of great importance in modern communication and space technology. Finally, possible extensions of the approach developed in the thesis are considered in the last chapter.

CHAPTER II

MATHEMATICAL BACKGROUND OF STOCHASTIC PROCESSES*

2-1 Introduction

The word stochastic is used to describe processes which are based on chance. The theory of stochastic processes in its complete generality requires the knowledge of the multidimensional probability distribution functions of a random variable and is usually too complicated and cumbersome for practical use. By restricting stochastic processes to have normal probability distribution functions, the knowledge of their first two moments is sufficient to determine all subsequent moments. The theory considering only the first two moments of a stochastic process is often refered to as the correlation theory of the stochastic processes. If the first two moments of a random input signal of a linear dynamical system are given, then one can always find the first two moments of the output of the system[10]. This, however, does not apply to the nonlinear systems where it is necessary to know the higher moments of the input stochastic signal.

2-2 Mathematical Description of a Stochastic Process

(A) Preliminary Discussion of a Random Variable

Consider a random experiment Z. Let A,B,... denote various

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* The basic material covered here can be found in the standard texts[8,9].

observable events associated with Z. Let Ω denote the sure event, that is an event which always occurs when Z is performed. Let ϕ denote an impossible event, an event which never occurs as an outcome of Z. Regard both Ω and ϕ as observable events. Let A^C denote the complementary event of A, the event that occurs if A fails to occur. Let $A \cup B$ be the event that at least A or B occurs. Let $A \cap B$ be the event that both A and B occur. Assume that A^C, $A \cup B$, $A \cap B$ are observable when A and B are observable. Then one says that the family of all observable events associated with Z constitutes a field F_o of events.

Let Ω be a space, the points ω of which may be entities of any kind. If E is any collection of ω sets, there is always a smallest field F_o including all the sets of E. A field F of ω sets is called a Borel field or a σ-field, if it includes all countable (finite or enumerable) unions and intersections of its sets. For any collection E of ω sets, there is always a smallest Borel field including E.

Assume that to every event A in the field F_o there corresponds a definite number P(A) called the probability of A possessing the properties $0 \leq P(A) \leq 1$, $P(\phi) = 0$, $P(\Omega) = 1$ and the further property of countable additivity: when $A = \cup_{i=1}^{\infty} A_i$ where $A, A_i \in F_o$ and $A_j \cap A_k = \phi$ when $j \neq k$, then $P(A) = \Sigma_{i=1}^{\infty} P(A_i)$. The set function P(A) is then a probability measure defined on the sets of the field F_o.

Halmos[11] states that there is a uniquely determined extension of P(A) to a probability measure defined on all sets A of the smallest σ-field F including F_o. A final extension is obtained by completing the σ-field F with respect to the measure P. This completed σ-field has the property that if it includes the set A for which P(A) = 0, it

also includes every subset A_1 of A, which will then have a measure $P(A_1) = 0$. The extension of P to this completed σ-field is still unique, and is called a complete probability measure.

A space Ω of points ω with a σ-field F of sets in Ω and a probability measure P defined on F constitutes a probability space (Ω,F,P). The sets of F are called measureable and P is said to define a probability distribution on Ω.

Consider a function $\xi = \xi(\omega)$ defined on Ω. ξ is said to be a random variable or measureable with respect to F, if it is a real-valued function and if, for every real number λ, $\{\omega \in \Omega;\ \xi(\omega) \leqslant \lambda\} \in F$. If ξ is a complex-valued function, ξ is said to be a complex-random variable if its real and imaginary parts are random variables. Two random variables are called equivalent if they differ on a set of measure zero.

For a random variable ξ, the function $G(\lambda) = P\{\xi(\omega) \leqslant \lambda\}$ is defined for all real λ and is called the distribution function of ξ. It is monotone non-decreasing, continuous on the right, and

$$\lim_{\lambda\to-\infty} G(\lambda) = 0, \quad \lim_{\lambda\to\infty} G(\lambda) = 1.$$

If $G(\lambda)$ is absolutely continuous, then $g(\lambda) = G'(\lambda)$ is called the probability density function corresponding to the distribution function $G(\lambda)$.

Expectation or mean value of a function $f(\xi)$ of the random variable ξ is defined as the Stieltjes integral

$$E\,\{f(\xi)\} = \int_\Omega f(\xi)dP = \int_{-\infty}^{+\infty} f(x)dG(x)$$

$E\,\xi^n$ is the n^{th} order moment of ξ

$E\,\xi \;\underline{\Delta}\; m$ — mean

$E\,(\xi - m)^2 = \sigma^2$ — variance, where σ is called the standard - deviation of ξ

If $E\,\{\xi\} < \infty$, then ξ is said to be integrable.

Normal Distribution

Let

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-y^2}{2}}\, dy.$$

Then

$$\phi(x) = \Phi'(x)$$
$$= \frac{1}{\sqrt{2\pi}}\, e^{\frac{-x^2}{2}}.$$

A random variable ξ is said to be normal (m,σ) if it has the distribution function (d.f.) $\Phi(\frac{x - m}{\sigma})$ and the density function $\frac{1}{\sigma}\,\phi\,(\frac{x - m}{\sigma})$. The distribution is symmetric about m and

$$E\,(\xi - m)^n = 0 \qquad \text{for } n \text{ odd}$$
$$= 1 \cdot 3 \cdot \cdot \cdot (2k - 1)\sigma^{2k} \qquad \text{for } n = 2k.$$

(B) Definition of Stochastic Process and Related Concepts

Let a probability space (Ω,F,P) and a parameter set T be given. A stochastic process is then defined as an indexed family of random variables $\{\xi(t,\omega)\}$, which for each fixed $t \in T$ is a measureable function of $\omega \in \Omega$ and for each fixed $\omega \in \Omega$ is an ordinary function of $t \in T$.

$\xi(\cdot,\omega)$ for a fixed ω describes the development (in time, space etc.) of the process and is a particular realisation or a sample function of the stochastic process. A sample function can be regarded as a point in the space X of all real or complex valued functions x of the

variable $t \varepsilon T$. The space X is called the sample (function) space of the stochastic process $\{\xi(t,\omega)\}$.

The process $\{\xi(t,\omega)\}$ thus determines a function which maps the space Ω onto a certain subspace of X and induces a probability distribution in X. A set A' in X is a family of functions of t on T. Every set $A' \varepsilon X$ has as its inverse image a certain ω set

$$A = \{\omega \varepsilon \Omega;\ \xi(\cdot,\omega) \varepsilon A'\}.$$

The family of such sets A' whose inverse images A are measureable sets forms a σ-field F'. The probability measure P' thus induced on X is defined for all sets $A' \varepsilon F'$ by $P'(A') = P(A)$. The triple (X,F',P') is a new probability space corresponding to the induced measure P'. We may regard ξ as a random function, taking values in X in accordance with the probability measure P'.

For an arbitrary finite set of t values, say $t_1,\ldots,t_n$, the corresponding random variables $\xi(t_1),\ldots,\ (t_n)$, will have a joint n-dimensional distribution with the d.f.

$$G(x_1,\ldots,x_n;\ t_1,\ldots,t_n) = P\{\xi(t_1) \leq x_1,\ldots,\xi(t_n) \leq x_n\}.$$

The family of all these joint probability distributions for $n = 1,2,\ldots$ and all possible values of t_j constitutes the family of finite dimensional distributions associated with the process $\{\xi(t,\omega)\}$.

An important question is how far is the probability distribution in the sample space X induced by a given process determined by the finite dimensional distribution of the process. An open interval in X is the set of real-valued function $x(t)$, which satisfy a finite set of inequalities of the form

$$a_j < x(t_j) < b_j \qquad (j = 1,2,\ldots,n)$$

where n is an arbitrary integer, $t_j \in T$, while a_j and b_j are real, finite or infinite. All finite unions of intervals form a field B_o of sets in X. Let B denote the smallest σ-field containing B_o. The sets of B are called the Borel sets of space X. The Kolmogorov Theorem[12] states that the family of finite dimensional distributions of any given stochastic process uniquely defines the probability distribution in the sample space X for all the Borel sets of the sample space X.

Further, given a family of finite dimensional distributions with the parameter set T, the necessary and sufficient condition for the existence of a stochastic process associated with these distributions is that the given family of distributions satisfy the following conditions of symmetry and consistency:

(i) The symmetry condition requires that the n^{th} dimensional d.f.

$$G(x_1,\ldots,x_n;\ t_1,\ldots,t_n) = P\{\xi(t_1) \leq x_1,\ldots,\xi(t_n) \leq x_n\}$$

be symmetric in all pairs (x_j,t_j) for an arbitrary finite set of t values $t_1,t_2,\ldots,t_n$, so that G remains invariant under any pairwise permutation of the pairs $\{x_j,t_j\}_{j=1}^{n}$.

(ii) The consistency condition is expressed by the relation

$$\lim_{x_{n+1},\ldots,x_{n+k} \to \infty} G(x_1,\ldots,x_n,\ldots,x_{n+k};\ t_1,\ldots,t_n,\ldots t_{n+k})$$
$$= G(x_1,\ldots,x_n;\ t_1,\ldots,t_n)$$

for every pair of integers n and $k \geq 1$.

2-3 Stochastic Processes With a Continuous Real Valued Parameter

Let T be the interval $[0,\infty)$ on the real line R. The probabili-

ties of many important events associated with a continuous parameter stochastic process, e.g. the probability that $\xi(\cdot,\omega)$ is continuous differentiable or integrable on some interval $I = [a,b]$ are not uniquely determined by the finite dimensional distributions of the process. Events of this type are not Borel sets because they involve the behaviour of ξ on an uncountable set of values of t. This is a great limitation since in applied situations, often something is known about the finite dimensional distributions and the objective is to get maximum information about the corresponding probability distribution in the sample function space.

The concept of separability introduced by Doob[8] imposes restrictions on sets determined by an uncountable number of random variables and allows one to determine their probabilities uniquely by finite dimensional distributions. The stochastic process

$$\xi = \{\xi(t,\omega),\ t \in T,\ \omega \in \Omega\}$$

is said to be separable (relative to sets closed in the sense of an appropriate topology on X) if there is a sequence $S = (s_i)$ of parameter values and a set $N \subset \Omega$ of probability zero such that for any closed set C in X and any open interval I on a real line, sets

$$\{\xi(t,\omega) \in C,\ t \in IT\} \subset \{\xi(s_j,\omega) \in C,\ s_j \in IS\}$$

differ by at most a subset of N. The set on the left which is determined by a noncountable number of coordinates, is assigned the same probability as the set on the right which is determined by a countable number of coordinates. It can be shown[13] that for any stochastic process $\{\xi(t,\omega\}$ there exists a stochastic process $\{\xi_o(t,\omega)\}$ defined on the same space Ω, separable relative to the class of closed sets, and

equivalent to $\{\xi(t,\omega)\}$, that is

$$P\{\xi(t) = \xi_o(t)\} = 1$$

for every fixed $t \varepsilon T$.

2-4 Gaussian (or Normal) Stochastic Process

A stochastic process $\{\xi(t,\omega),\ t \varepsilon T\}$ is called Gaussian if the joint distribution of every finite set of ξ_t's is Gaussian. A general Gaussian process is determined as follows[14].

Define a projection operator $P_n = P_{t_1 \ldots t_n}$ on X, the linear space of functions over T as

$$P_{t_1 \ldots t_n}[x(t)] = \{x(t_1),\ldots,x(t_n)\},\ t_1,\ldots,t_n \varepsilon T,\ x \varepsilon X.$$

Let R_{t_i} be the family of real lines $(i = 1,2,\ldots,n)$ and B_{t_i} be the corresponding family of Borel sets. Then an n dimensional Borel measureable space (Ω_{T_n}, B_{T_n}) is defined by the Cartesian product of the sequence $\{R_{t_i}, B_{t_i}\}$ of Borel lines and is given by

$$(\Omega_{T_n}, B_{T_n}) = \prod_{i=1}^{n} (R_{t_i}, B_{t_i}).$$

Clearly t plays only the role of a parameter and

$$P_{t_1 \ldots t_n}\{x(\cdot)\} = (x_{t_1};\ \ldots;\ x_{t_n}).$$

The value of x_{t_i} depends on the element $x(\cdot) \varepsilon X$. Let us introduce a parameter σ to explicitly indicate this dependence. Thus

$$P_{t_1 \ldots t_n}\{x(\cdot)\} = \{x_{t_1}(\sigma);\ \ldots;\ x_{t_n}(\sigma)\}$$

where

$$\sigma \varepsilon B_{T_n}.$$

B_{T_n} may be generated by a class C_{T_n} of sets of the form

$$\{C_{T_n}{}^k\} = \{a_1{}^k,\ldots,a_n{}^k;\ b_1{}^k,\ldots,b_n{}^k\}$$

where

$$A_k \underline{\Delta} \{a_1{}^k,\ldots,a_n{}^k;\ b_1{}^k,\ldots,b_n{}^k\}$$
$$= \{\sigma \in X;\ a_i{}^k < \sigma_i \le b_i{}^k,\ 1,2,\ldots,n\}.$$

B_{T_n} is said to be generated by the class C_{T_n} of n dimensioanl cells in Ω_{T_n}.

If the linear space X is known to possess a complete orthonormal basis $\{\phi_n\}$, the projection P_n can be taken as the usual orthogonal projection of X into its n dimensional subspace. Then

$$P_n x(\cdot) = \left(<x,\phi_1>(\sigma),\ldots,<x,\phi_n>(\sigma)\right).$$

Let $A_{T_n}{}^k$ be an element of B_{T_n} (the class of Borel sets in Ω_{T_n}). Then a Borel cylinder is defined in X by $\{A_{T_n}{}^k \times \Omega_{T_n}{}^c\}_k$ for any finite n with $A_{T_n}{}^k$ as the base. For a fixed n, the cylinder so defined from a σ-ring $\delta_{\Omega T_n}$. Kolmogorov's consistency relation demands that

$$P_{B_T}(A_{T_n}{}^k \times \Omega_{T_n}{}^c) = P_{B_{T_n}}(A_{T_n}{}^k).$$

Now a stochastic process $\{x(t),\ t \in T\}$ is said to be a Gaussian process if for any $P_n = P_{t_1 \ldots t_n}$, $n < \infty$, the vector $\underline{x}_n = (x_{t_1},\ldots,x_{t_n})$ has the probability density function

$$g(\underline{x}_n) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \frac{1}{\sqrt{|M_n|}} \exp-\frac{1}{2}(\underline{x}_n - A_n)^T M_n{}^{-1}(\underline{x}_n - A_n)$$

where

$A_n = E\,\underline{x}_n = (a_1,\ldots,a_n)$, $|M_n|$ is the determinant of the matrix M_n and M_n is the $n \times n$ positive definite moment matrix, viz.

$$M_n = \begin{bmatrix} E\,(x_{t_1},x_{t}) & \cdots & E\,(x_{t_1},x_{t_n}) \\ \vdots & & \vdots \\ E\,(x_{t_n},x_{t_1}) & \cdots & E\,(x_{t_n},x_{t_n}) \end{bmatrix}$$

It can be shown that the probability measure $P_{B_{T_n}}$ defined by

$$dP_{B_{T_n}}(\underline{x}_n) = [(\frac{1}{2\pi})^{\frac{n}{2}}\frac{1}{\sqrt{|M_n|}} \exp-\frac{1}{2}(\underline{x}_n - A_n)^T M_n^{-1}(\underline{x}_n - A_n)]dx_1 \ldots dx_n$$

is a countably additive measure function for any finite n. Here the sequence $\{x_n\}$ may be either $\{\langle x,\phi_n\rangle\}$ or $\{x_{t_n}\}$ and all finite dimensional measure functions $P_{B_{T_n}}$ obtained from some suitable family of all distributions define the measure of all finite dimensional Borel cylinders in X through the Kolmogorov consistency relation.

If $\{\eta(t),\ t \in T\}$ be any stochastic process with

$$E\ \{|\eta_t|^2\} < \infty,\quad t \in T,$$

then there exists a corresponding Gaussian process $\{\xi(t)\}$ with the same parameter set but defined in a different measure space, whose random variables ξ_t satisfy

$$E\ \{\xi_t\} = 0$$

$$E\ \{\xi_s\overline{\xi}_t\} = E\ \{\eta_s\overline{\eta}_t\} \qquad s,t \in T.$$

However, very few facts specifically true of Gaussian processes are known and a further structure has to be put on the stochastic process to carry out the analysis.

2-5 Strictly Stationary and Stationary Gaussian Processes

A stochastic process $\{\xi(t)\}$ is said to be strictly stationary if for any finite sequence of parameter points $t_1,\ldots,t_n$, the joint distribution of n complex random variables $\xi(t_1 + \tau),\ldots,\xi(t_n + \tau)$ should be independent of τ. Thus

$$E\ \xi(t) = E\ \xi(t + \tau). \tag{2.1}$$

Without loss of generality it can be assumed that $E\ \xi(t)$ is zero.

The covariance function of a stochastic process ξ is defined as $r(t,u) = E\ \xi(t)\overline{\xi(u)}$ where $\overline{\xi(u)}$ is the complex conjugate of $\xi(u)$. If ξ is a strictly stationary process, then for any t, u and $\tau \varepsilon T$

$$\begin{aligned} r(t,u) &= E\ \xi(t)\overline{\xi(u)} \\ &= E\ \xi(t+\tau)\overline{\xi(u+\tau)} \\ &= E\ \xi(t-u)\overline{\xi(0)} \qquad (\text{putting } \tau = -u). \end{aligned}$$

This shows that $r(t,u)$ is a function only of the difference $t - u$.

The class of all stochastic processes with finite second moments satisfying

$$\left.\begin{aligned} &E\ \xi(t) = m \qquad (\text{usually } m \text{ is taken to be zero}) \\ &E\ \xi(t)\overline{\xi(u)} = r(t-u) \end{aligned}\right] \quad 2.2$$

is called the class of stationary processes.

A stationary process will not necessarily be strictly stationary because the relation 2.2 does not imply the invariance of all the joint distributions of the variables. A strictly stationary process will not satisfy 2.2 since the moments of the first and second order may not exist. However, the class of strictly stationary processes with finite second order moments forms a subclass of all stationary processes. For any Gaussian process, all moments are finite. Thus a strictly stationary Gaussian process is always stationary. However, in order that a Gaussian stationary process $\{\xi(t)\}$ with $E\ \xi(t) = 0$ should be strictly stationary, it is necessary and sufficient that the covariance moment

$$q(t,u) = E\ \xi(t)\xi(u)$$

formed without taking the complex conjugate of the second factor,

should be a function of $(t - u)$.[9] For a real Gaussian process, stationarity and strict stationarity are equivalent.

A notion of convergence of random variables is necessary for further discussion. Suppose a sequence of random variables $\xi_1, \ldots, \xi_n, \ldots$ is given, all the $\xi_n = \xi_n(\omega)$ being defined on the same probability space. Let ξ be another random variable defined on the same probability space. There are three principle modes of convergence of ξ_n to the limit ξ as $n \to \infty$.

(i) ξ_n converges to ξ almost everywhere (a.e.) if $P(\xi_n \to \xi) = 1$. Notation -- $\xi_n \xrightarrow{\text{a.e.}} \xi$.

(ii) ξ_n converges to ξ in quadratic mean (q.m.) or in the mean square if $E\,|\xi_n - \xi|^2 \to 0$. Notation — $\xi_n \xrightarrow{\text{q.m.}} \xi$.

(iii) ξ_n converges to ξ in probability or in measure if for every $\varepsilon > 0$, $P(|\xi_n - \xi| > \varepsilon) \to 0$. Notation — $\xi_n \xrightarrow{\text{pr}} \xi$.

The natural convergence concept in connection with processes with finite second order moments is convergence in quadratic mean. If the random variables $\xi(t)$ satisfy the condition $\xi(t) \xrightarrow{\text{q.m.}} \xi(t_o)$ as $t \to t_o$, then $\xi(t)$ is said to be continuous in the quadratic mean at $t = t_o$. $\xi(t)$ is continuous in the quadratic mean, if and only if the covariance function $r(t,u)$ is continuous at the point $t = u = t_o$. For any stationary process $\{\xi(t)\}$

$$E\,\xi(t)\overline{\xi(u)} = r(t - u)$$

$$E\,\xi(u)\overline{\xi(t)} = r(u - t)$$

so that the covariance function satisfies the relation

$$r(-t) = \overline{r(t)}.$$

The variance is independent of t, since for every t

$$E\ |\xi(t)|^2 = r(0).$$

Further

$$|r(t)| \leq r(0).$$

To exclude trivial cases, it is always assumed that $r(0) > 0$. Hence a stationary process $\xi(t)$ is continuous in the q.m. for every t, if and only if $r(t)$ is continuous at the point $t = 0$ which in turn also implies that $r(t)$ is continuous everywhere. In the following discussion it is assumed that this continuity condition is always satisfied for a stationary process.

A covariance function $r(t)$ of a stationary process $\{\xi(t)\}$ can be represented in the form

$$r(t) = \int_{-\infty}^{+\infty} e^{it\lambda} dG(\lambda) \qquad 2.3$$

where $G(\lambda)$ is a real, non-decreasing and bounded function. The representation 2.3 can be regarded as a spectral representation of $r(t)$. $G(\lambda)$ is called the spectral distribution function of the process $\{\xi(t)\}$ and if it is absolutely continuous, the derivative $g(\lambda) = G'(\lambda)$ is called the spectral density of the process. For a real stationary process

$$r(t) = \int_0^{\infty} \cos \lambda t dG(\lambda).$$

It is important to have information about analytic properties like continuity or differentiability of sample functions. Dorbushin[15] has shown that a stationary Gaussian process has, with probability one, sample functions which are either continuous everywhere or else have a discontinuity of the second kind at every point. Belayev[16] showed that if sample functions are not continuous, they are unbounded with probability one in any finite interval.

Hunt[17] has given sufficient conditions for continuity of sample functions in terms of the spectrum of the stationary Gaussian process. Let $\xi(t)$ be a stationary process with spectral density distribution $G(\lambda)$. If

$$\int_0^{\infty}[\ln(1+\lambda)]^a dG(\lambda) < \infty$$

for some $a > 1$, then there is an equivalent process $\{\eta(t)\}$ possessing continuous sample functions, with probability one. If instead of the above for some $a > 1$

$$\int_0^{\infty}\lambda^2[\ln(1+\lambda)]^a dG(\lambda)$$

then there is an equivalent process $\{\eta(t)\}$ which has a continuous sample derivative, with probability one. Belayev[16] translated the above result into a condition on the covariance function, viz. if the covariance function $r(t)$ of the process $\{\xi(t)\}$ is such that

$$E\,|\xi(t+h) - \xi(t)|^2 \leq \frac{C}{|\ln|h||^a}$$

for some $a > 1$, $C > 0$ and for all suficiently small h, then there exists an equivalent process $\{\eta(t)\}$ whose sample functions are continuous with probability one.

2-6 Ergodicity

The ergodic hypothesis provides a link between the time average based on one single realisation of the process and the ensemble average extended over the set of all possible sample functions of the process.

Let $\{\xi(t)\}$ be a given real valued Gaussian stationary process which is always strictly stationary. Consider the probability space (X,B,Π) where X is the space of all real valued functions on T, B is the smallest σ-field over the intervals of X and Π is the

probability measure uniquely determined on all sets of B by the finite dimensional distributions of $\xi(t)$. Now define a shift transformation U_τ in the probability space taking $\omega = x(t)$ into $\omega_\tau = x(t - \tau)$. The transformation U_τ for all real τ form a group and $U_{\tau+\rho} = U_\tau U_\rho$. The transformations U_τ are measure preserving, because the finite dimensional distributions defining Π are strictly stationary.

Thus U_τ takes any set $S \varepsilon B$ $\{$of functions $x(t)\}$ into the set S_τ formed by the shifted functions $x(t + \tau)$ and $\Pi(S_\tau) = \Pi(S)$ for all real τ. A Borel set S is called an invariant set of the process $\{\xi(t)\}$, if for every fixed τ, the sets S and S_τ differ, at most, by sets of Π-measure zero. The invariant sets form a σ-field contained in B. All sets of probability $\Pi = 0$ and $\Pi = 1$ are invariant.

The strictly stationary process $\xi(t)$ is called ergodic (or metrically transitive) if the σ-field of invariant sets only contains sets of probability zero or one.

Any random variable η defined by means of the random variables $\xi(t)$ for any value of t is called a random variable defined on $\{\xi(t)\}$. $\eta(t)$ is said to be an invariant random variable of the process $\{\xi(t)\}$, if for every fixed τ, the random variables η and $U_\tau\eta$ are equivalent.

Now assume that the strictly stationary process $\xi(t)$ satisfies the two additional conditions:

(i) $E\ |\xi(t)| < \infty$

(ii) With probability one, the sample functions of the $\xi(t)$ process are Riemann integrable on every finite interval. Then each of the time averages

$$\frac{1}{T}\int_{-T}^{0}\xi(t)dt \qquad \text{and} \qquad \frac{1}{T}\int_{0}^{T}\xi(t)dt$$

converges with probability one to an invariant random variable of the $\xi(t)$ process, as $T \to \infty$. If, in particular, $\xi(t)$ is ergodic, both limits are, with probability one equal to the constant $E\,\xi(0)$. Thus

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{0} \xi(t)dt = E\,\xi(0)$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} \xi(t)dt.$$

Here $E\,\xi(0)$ is an average value of the random variable $\xi(0)$, over the set of all possible sample functions of the process, whereas $\lim_{T\to\infty} \frac{1}{T}\int_{-T}^{0} \xi(t)dt$ is the limit of the time average, extended over the values assumed in the past by one single realisation of the process.

When the process $\xi(t)$ is real valued, Gaussian and stationary, and with probability one its sample functions are continuous over any finite interval; then conditions (i) and (ii) are satisfied and so $\xi(t)$ is ergodic. In terms of spectral representation, Grenander[18] has shown that $\xi(t)$ is ergodic if and only if $H(\lambda) = 2G(\lambda) - r(0)$ is everywhere continuous where $G(\lambda)$ is the spectral distribution of the process $\xi(t)$.

In applied situations, $\phi_{\xi\xi}(\tau)$ defined by

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} \xi(t)\xi(t+\tau)d\tau$$

is called the autocorrelation function. Under the assumption of stationarity and ergodicity the covariance function

$$r(\tau) = \phi_{\xi\xi}(\tau).$$

2-7 Random Fixed Point Theorems

The systematic study of random equations, employing the methods

of functional analysis was intitiated by Špaček[19]. The essential feature lies in the introduction of the notion of the "generalized random variable". Then most of the notions of functional analysis are carried over into the probabilistic functional analysis "in probability".

Let (Ω,A,μ) denote a complete probability measure space. Let (X,T) be a measureable space where X is a complex, separable Banach space, and B is the σ-algebra of all Borel subsets of X.

A mapping $x(\omega)$ of $\Omega \to X$ is called a generalized random variable (with values in Banach space X) if

$$\{\omega;\ x(\omega) \in B\} \in A$$

for all $B \in T$.

A mapping $T(\omega)$ of Cartesian product space $\Omega \times X$ into a Banach space Y is called a random transformation if $T(\omega)[x]$ is, for $x \in X$, a generalized random variable with values in Y.

A random transformation $T(\omega)$ is said to be

(i) linear, if

$$T(\omega)[\alpha x_1 + \beta x_2] = \alpha T(\omega)[x_1] + \beta T(\omega)[x_2]$$

for every $\omega \in \Omega$, $x_1,x_2 \in X$ and $\alpha,\beta \in R$.

(ii) bounded, if there exists a mapping $c(\omega)$ on Ω into R (i.e. $c(\omega)$ is a real valued random variable) such that for all $\omega \in \Omega$ and $x \in X$,

$$||T(\omega)x|| \leq c(\omega)||x||.$$

A random transformation $T(\omega)$ on $\Omega \times X$ into X is called a random operator.

The mapping $S(\omega)$ is said to be the inverse of the random operator $T(\omega)$ if

$$\mu\{\omega;\ T(\omega)[S(\omega)x] = x \qquad \text{for every } x \in X\} = 1.$$

The following results are due to Hans[20].

(i) If $x_1(\omega), x_2(\omega), \ldots$ is a sequence of generalized random variables with values in X converging almost surely to a mapping $x(\omega)$ of Ω into X, then $x(\omega)$ is a generalized random variable with values in X.

(ii) Let $x(\omega)$ be a generalized random variable with values in X, and let $T(\omega)$ be a continuous random operator on $\Omega \times X$ to X. Then the mapping $y(\omega)$ of Ω into X defined for every $\omega \in \Omega$ by

$$y(\omega) = T(\omega)[x(\omega)]$$

is a generalized random variable with values in X.

Random Equations

Consider the deterministic operator equation

$$T[\xi] = y \tag{2.5}$$

where

$$\xi, y \in X \text{ and } T\colon X \to X.$$

Let

$$S = \{x \in X;\ T[x] = y\}.$$

S is called the solution set of Equation 2.5. If $S \neq 0$, then Equation 2.5 is said to be solvable and if S has only one element, we say that Equation 2.5 has a unique solution. Consider stochastic analogues of Equation 2.5.

(i) $y = y(\omega)$ can be a generalized random variable with values in X. In case T^{-1} exists, then $x(\omega) = T^{-1}[y(\omega)]$ will also be a generalized random variable with values in X.

(ii) $T = T(\omega)$ can be a random operator on $\Omega \times X \to X$, and y can

be a deterministic element. Then

$$T(\omega)[x] = y$$

is called a random operator equation and $x(\omega) = T^{-1}(\omega)[y]$ will be, for the set of ω's for which $T^{-1}(\omega)$ exists, a generalized random variable with values in X.

(iii) In the most general case $T(\omega)$ is a random operator on $\Omega \times X$ to X and $y = y(\omega)$ is a generalized random variable with values in X. Hence Equation 2.5 becomes

$$T(\omega)[x] = y(\omega) \qquad 2.6$$

$$x(\omega) = T^{-1}(\omega)[y(\omega)].$$

In this thesis, we will be concerned only with case (i) but the following discussion holds good for case (iii) as well.

Every mapping $x(\omega)$ of Ω into X satisfying

$$T(\omega)[x(\omega)] = y(\omega)$$

for every $\omega \in \Omega_o$, where $\mu(\Omega_o) = 1$, is said to be a wide sense solution of Equation 2.6. Every generalized random variable $x(\omega)$ with values in X satisfying the condition

$$\mu\{\omega;\ T(\omega)[x(\omega)] = y(\omega)\} = 1$$

is said to be a random solution of the random operator Equation 2.6.

In order to study the existence, uniqueness and measureability of the random solutions of random operator equations, use is made of probabilistic analogues of the results in the theory of the deterministic operator equations. There the principle of contraction mapping and fixed point theorems play a very important role[21] and are briefly described below.

Let X be an arbitrary metric space with distance function

$d(x_1,x_2)$. A mapping T of X into itself is said to be a contraction mapping if there exists a number $c < 1$ such that $d(Tx_1,Tx_2) \leq cd(x_1,x_2)$ for any two elements $x_1,x_2 \in X$. Every contraction mapping defined on a complete metric space X has one and only one fixed point.

Similarly, random operator $T(\omega)$ on $\Omega \times X$ to X is said to be a random contraction operator if there exists a real valued random variable $c(\omega)$ such that $c(\omega) < 1$ for all $\omega \in \Omega$ and such that

$$||T(\omega)[x_1] - T(\omega)[x_2]|| \leq c(\omega)||x_1 - x_2||$$

for every $\omega \in \Omega$ and for every pair of generalized random variables $x_1,x_2 \in X$. If $c(\omega) = c < 1$ for all $\omega \in \Omega$, $T(\omega)$ is said to be a uniform random contraction operator. The stochastic analogue of Banach fixed point theorem due to Hans[20] is as follows:

Let $T(\omega)$ be a random contraction operator on $\Omega \times X$ to X. Then there exists a generalized random variable $\xi(\omega)$ with values in X such that

$$\mu\{\omega\colon T(\omega)[\xi(\omega)] = \xi(\omega)\} = 1. \qquad 2.7$$

The generalized random variable $\xi(\omega)$ is unique in the sense that if $\psi(\omega)$ is another generalized random variable satisfying $T\xi = \xi$, then

$$\mu\{\omega\colon \xi(\omega) = \psi(\omega)\} = 1.$$

Further the random fixed point $\xi(\omega)$ can be obtained by successive approximations starting from an arbitrary generalized random variable $x_o(\omega) \in X$.

2-8 Class of Systems Considered in This Thesis

We are concerned with the random operator of the type

$$T[x(\omega)] = y(\omega)$$

where $y(\omega)$ is a generalized random variable with values in X. Further, we are interested in the situation where T is a uniform contraction operator and the domain of T is the space of stationary Gaussian ergodic processes with finite second moments. The range space is a linear complete metric space[22] with $d(\xi,\eta) = E\,|\xi - \eta|$ and becomes a Banach space with norm $||\xi|| = E\,|\xi|$, provided the equivalent random variables are identified to be the same.

CHAPTER III

ANALYSIS OF A NONLINEAR FEEDBACK SYSTEM WITH A SQUARE NONLINEARITY

3-1 Introduction

Consider the following system

$$L(x_1) + \alpha\phi(x_1) = y(t) \qquad 3.1$$

where L is a linear, time invariant differential operator with zeros only in the left half plane, α is a constant, $\phi(x_1)$ is a nonlinear function of x_1 and y(t) is the forcing function. Fig. 3.1 is the block diagram representation of system 3.1.

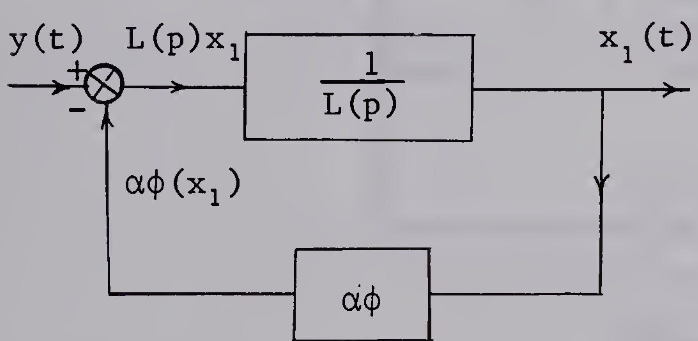
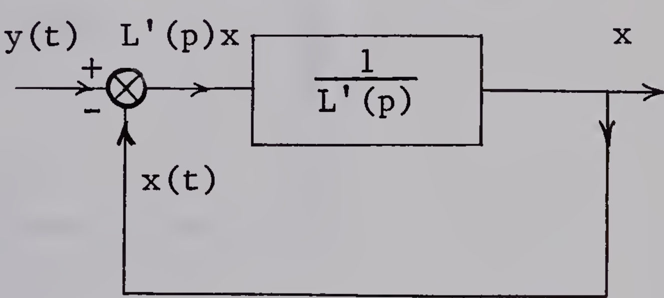


Figure 3.1



$L'(p) + 1 = L(p)$

Figure 3.2

The nonlinear system can be considered as a perturbed version of the following linear system (see Fig. 3.2).

$$L(x) = y(t) \qquad 3.2$$

where $x_1 = x + \mu$. This is basically the approach considered by Christensen[23], the objective being to obtain a recursion relation

$$\mu = A(\mu, y) \qquad 3.3$$

subject to the constraint that A be a contraction operator. y(t) is taken to be an arbitrary, bounded driving function. Thus through the use of relation 3.3, a region can be found where system 3.1 gives a

unique bounded output for each bounded input. In the following discussion, a similar approach is formulated for the case when y(t) is a stochastic process under suitable restrictions.

3-2 A Feedback System With Square Nonlinearity

Consider the system shown in Fig. 3.3,

$$L(x_1) + \alpha(x_1^2) = y(t) \qquad 3.4$$

where L(p) is a linear, time-invariant differential operator with zeros only in the left half plane and y(t) is a sample function from a real,

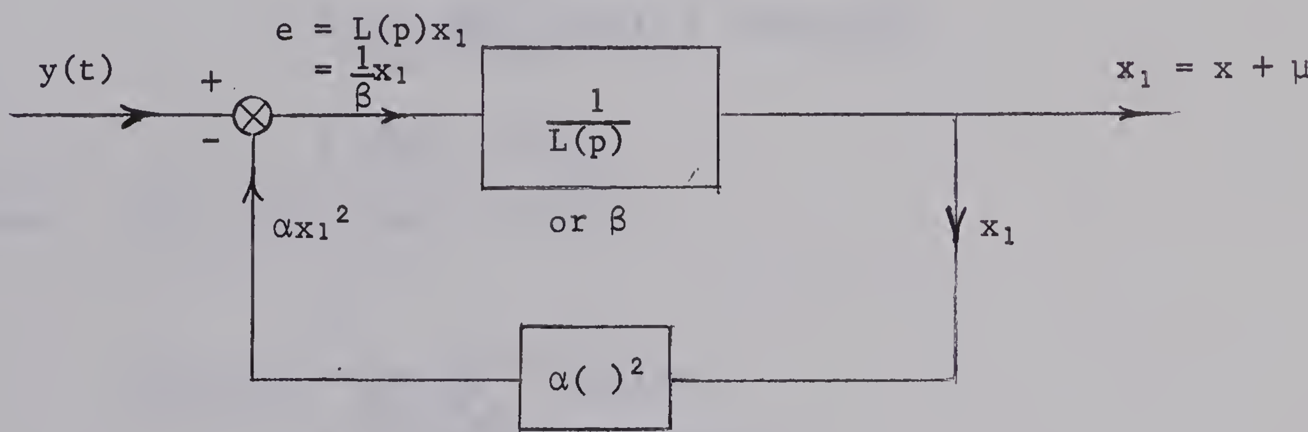
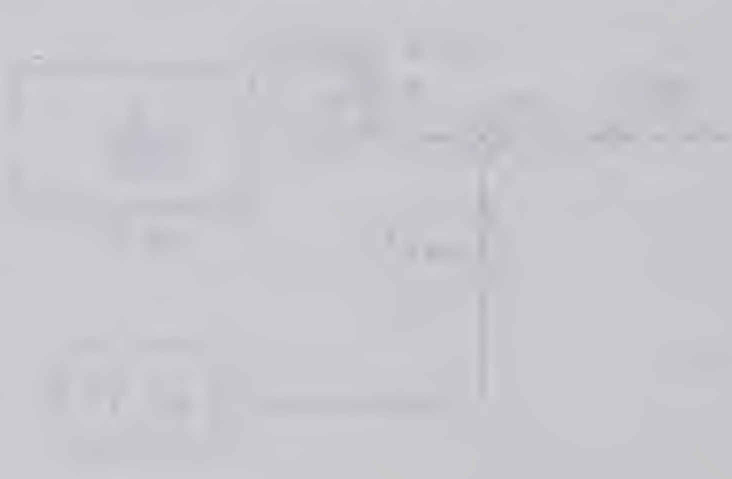


Figure 3.3

stationary, ergodic Gaussian process. The corresponding unperturbed linear system is L(x) = y(t). Let

$$x_1(t) = x(t) + \mu(t). \qquad 3.5$$

A simplifying assumption is made that μ(t) is also a stationary Gaussian process. This can be partially justified by assuming that the input process has sample functions which are continuous everywhere with probability one. The discussion of the results of Dorbushin and Belayev in Chapter II shows that this is not an unduly restrictive constraint. Hence $x_1(t)$ is also a stationary, ergodic Gaussian process and hence so is e(t). Thus we are always operating within the domain space X of



stationary, ergodic Gaussian processes.

Introduce a norm by the relation

$$||x(t)|| = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} |x(t)|dt. \qquad 3.6$$

This relation must satisfy the axioms of the norm function viz.

(i) $||x|| \geq 0$ and $||x|| = 0$ if and only if $x = 0$.

(ii) $||\lambda x|| = |\lambda|\ ||x||$

(iii) $||x + y|| \leq ||x|| + ||y||$.

Now

$$||x + y|| = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} |x(t) + y(t)|dt$$

$$\leq \lim_{T\to\infty} \frac{1}{2T}\left[\int_{-T}^{T} [|x(t)| + |y(t)|]dt\right]$$

$$= ||x|| + ||y||.$$

Therefore $||x + y|| \leq ||x|| + ||y||$.

Also

$$||\lambda x(t)|| = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} |\lambda x(t)|dt$$
$$= |\lambda| \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} |x(t)|dt$$
$$= |\lambda|\ ||x(t)||.$$

The defining relation

$$||x(t)|| = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} |x(t)|dt$$

implies $||x|| = 0$ if and only if $x = 0$. Hence the relation defined by 3.6 satisfies the axioms of the norm.

Consider the operator equation $L(x) = y(t)$, or

$$x(t) = \beta(y)$$
$$= \int_0^\infty h(t - \tau)y(\tau)d\tau$$

where $h(t)$ is the impulse response of the operator β and $h(t) = \mathcal{L}^{-1}\frac{1}{L(p)}$. Youla[24] has shown that a necessary and sufficient condition that a bounded input to a linear, time invariant system gives rise to a bounded

output is that

$$\int_0^{\infty} |h(t)|dt < \infty.$$

Let

$$\int_0^{\infty} |h(t)|dt = H.$$

In the representation shown in Fig. 3.4, $y_i(t)$ is the input to β and the output is $y_o(t)$. β can be considered to be a linear operator

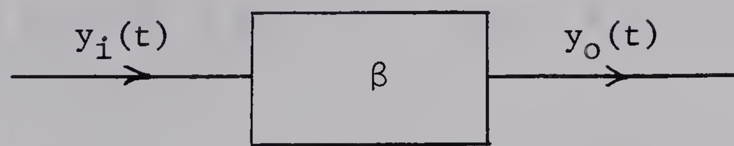


Figure 3.4

mapping the Banach space X into itself.

$$\begin{aligned} y_o(t) &= \beta\big(y_i(t)\big) \\ &= \int_0^{\infty} h(t - \tau)y_i(\tau)d\tau \end{aligned}$$

$$\begin{aligned} ||y_o(t)|| &\le ||\beta y_i|| \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \Big|\int_0^{\infty} h(t - \tau)y_i(\tau)d\tau\Big|dt \\ &\le \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \Big[\int_0^{\infty} |h(t - \tau)||y_i(\tau)|d\tau\Big]dt. \end{aligned}$$

Interchanging the order of integration

$$\begin{aligned} ||y_o(t)|| &\le \int_0^{\infty} \Big(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |h(t - \tau)||y_i(\tau)|dt\Big)d\tau \\ &= \int_0^{\infty} |h(\tau)|\Big(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} |y_i(t - \tau)|dt\Big)d\tau \\ &= \int_0^{\infty} |h(\tau)|d\tau||y_i|| \\ &= H||y_i||. \end{aligned}$$

Hence β is a linear bounded operator and so is continuous.

An operator A mapping an arbitrary metric space (Z,ρ) into itself is said to be a contraction operator[25] if for every x_1,x_2 in Z,

$$\rho(Ax_1,Ax_2) \le K\rho(x_1,x_2)$$

where K is a constant $0 \leq K < 1$.

Further, given a sphere S of radius γ in a Banach space X about x_o (that is $||x - x_o|| < \gamma$) and an operator A mapping S into X such that

$$||Ax_1 - Ax_2|| \leq K||x_1 - x_2||$$

for any pair of points x_1, x_2 of S, where $0 \leq K < 1$, then the operator equation $Ax = x$ has a unique solution x^* in S

$$x^* = x_o + (x_1 - x_o) + \ldots + (x_n - x_{n-1}) + \ldots$$

where

$$\begin{aligned} x_1 &= Ax_o \\ &\vdots \\ x_n &= Ax_{n-1} \\ &\vdots \end{aligned}$$

provided the following fixed point condition is satisfied

$$||Ax_o - x_o|| < \gamma(1 - K).$$

For the system 3.4, x^* is obtained by Volterra series through an iteration proceedure. Thus, a convergence criterion is formulated for the Volterra series so obtained, based on the contraction mapping principle. This approach is essentially that of Christensen[27].

Referring to Fig. 3.3,

$$e = y - \alpha x_1^2.$$

But $x_1 = \beta e$. Therefore

$$e = y - \alpha(\beta e)^2$$

$$e^2 = y^2 - 2\alpha y(\beta e)^2 + \alpha^2(\beta e)^4.$$

Putting $e^2 = z$, one gets

$$\begin{aligned} z &= y^2 - 2\alpha y(\beta\sqrt{z})^2 + \alpha^2(\beta\sqrt{z})^4 \\ &= A(z,y). \end{aligned} \qquad 3.7$$

For the contraction condition to hold,

$$||A(z_1,y) - A(z_2,y)|| \leq K||z_1 - z_2|| \qquad \text{where } 0 \leq K < 1$$

that is

$$||y^2 - 2\alpha y(\beta\sqrt{z_1})^2 + \alpha^2(\beta\sqrt{z_1})^4 - y^2 + 2\alpha y(\beta\sqrt{z_2})^2 - \alpha^2(\beta\sqrt{z_2})^4||$$
$$\leq K||z_1 - z_2||$$

or

$$||-2\alpha y(\beta\sqrt{z_1} - \beta\sqrt{z_2})(\beta\sqrt{z_1} + \beta\sqrt{z_2}) + \alpha^2(\beta\sqrt{z_1} - \beta\sqrt{z_2})(\beta\sqrt{z_1} + \beta\sqrt{z_2}) \circ [(\beta\sqrt{z_1})^2$$
$$+ (\beta\sqrt{z_2})^2]||$$
$$\leq K||(\sqrt{z_1} - \sqrt{z_2})(\sqrt{z_1} + \sqrt{z_2})||$$

or

$$||\beta^2|| \; ||\sqrt{z_1} - \sqrt{z_2}|| \; ||\sqrt{z_1} + \sqrt{z_2}|| \; ||\{\alpha^2[(\beta\sqrt{z_1})^2 + (\beta\sqrt{z_2})^2] - 2\alpha y\}||$$
$$\leq K||\sqrt{z_1} - \sqrt{z_2}|| \; ||\sqrt{z_1} + \sqrt{z_2}||$$

or

$$||\beta^2||\{\alpha^2||(\beta\sqrt{z_1})^2 + (\beta\sqrt{z_2})^2|| + 2|\alpha| \; ||y||\} \leq K.$$

Now let

$$||y|| = Y$$

$$||\beta|| = H$$

so, we get

$$\alpha^2H^2||(\beta\sqrt{z_1})^2 + (\beta\sqrt{z_2})^2|| \leq K - 2|\alpha|H^2Y$$

or

$$\alpha^2H^4[||\sqrt{z_1}||^2 + ||\sqrt{z_2}||^2] \leq K - 2|\alpha|H^2Y.$$

Let

$$||z|| \geq ||z_1||$$

$$||z|| \geq ||z_2||.$$

Then

$$\alpha^2H^4 2||z|| \leq K - 2|\alpha|H^2Y$$

or

$$||z|| \le \frac{K - 2|\alpha|H^2 Y}{2\alpha^2 H^4}$$

$$||z|| = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} e^2 dt$$
$$= \phi_{ee}. \qquad 3.8$$

Hence

$$\phi_{ee} \le \frac{K - 2|\alpha|H^2 Y}{2\alpha^2 H^4}.$$

Taking the first approximation $z_o = 0$, for the fixed point condition to be satisfied,

$$||A(z_o,y) - z_o|| < (1 - K)||z||$$

or

$$||y^2|| < (1 - K)||z||$$

or

$$\phi_{yy} < (1 - K)\phi_{ee}$$
$$\le (1 - K)\left[\frac{K - 2|\alpha|H^2 Y}{2\alpha^2 H^4}\right] \qquad 3.9$$

It is to be noted that Equation 3.9 includes both the contraction and the fixed point conditions, and puts a constraint on the autocorrelation ϕ_{yy} of the input.

Now considering Equation 3.7

$$z = y^2 - 2\alpha y(\beta\sqrt{z})^2 + \alpha^2(\beta\sqrt{z})^4$$

1st approximation

$$z_o = 0$$

2nd approximation

$$z_1 = y^2$$

3rd approximation

$$z_2 = y^2 - 2\alpha y(\beta y)^2 + \alpha^2(\beta y)^4$$

and so on

$$z = z_o + (z_1 - z_o) + (z_2 - z\ ) + \ldots + (z_n - z_{n-1}) + \ldots \qquad 3.10$$

Provided Equation 3.9 is satisfied, the iteration series given by Equation 3.10 will converge to the solution of Equation 3.7.

Another way to analyse the system 3.4 is by the following method

$$y(t) - \alpha x_1^2 = \frac{1}{\beta} x_1$$

or

$$x_1 = \beta y - \alpha\beta x_1^2$$

or

$$x + \mu = \beta y - \alpha\beta(x + \mu)^2$$

or

$$\beta y + \mu = \beta y - \alpha\beta(\beta y + \mu)^2$$

$$\mu = -\alpha\beta(\beta y + \mu)^2$$

$$\triangleq A(\mu,y) \qquad 3.11$$

where

$$A(\mu,y) = -\alpha\beta(\beta y + \mu)^2.$$

Taking

$$\mu_o = 0$$

$$\mu_1 = -\alpha\beta(\beta y)^2$$

$$\mu_2 = -\alpha\beta\{\beta y - \alpha\beta(\beta y)^2\}^2$$

$$\vdots$$

etc., and

$$\mu = \mu_o + (\mu_1 - \mu_o) + (\mu_2 - \mu_1) + \ldots + (\mu_n - \mu_{n-1}) + \ldots \qquad 3.12$$

it follows that the series 3.12 converegges to the solution of 3.11 if contraction and fixed point conditions are satisfied. For the contraction condition to hold

$$||A(\mu_1,y) - A(\mu_2,y)|| \leq K||\mu_1 - \mu_2|| \qquad \text{where } 0 \leq K < 1$$

Therefore

$$||-\alpha\beta(\beta y + \mu_1)^2 + \alpha\beta(\beta y + \mu_2)^2|| \leq K||\mu_1 - \mu_2||$$

or

$$|\alpha| \; ||\beta|| \; ||(\beta y + \mu_2)^2 - (\beta y + \mu_1)^2|| \leq K||\mu_1 - \mu_2||$$

or

$$|\alpha| \; ||\beta|| \; ||[\mu_2 - \mu_1][(\beta y + \mu_2) + (\beta y + \mu_1)]|| \leq K||\mu_1 - \mu_2||$$

or

$$2|\alpha| \; ||\beta||[||\beta y|| + ||\mu||] \leq K$$

because

$$||\mu|| \geq ||\mu_1||$$

and

$$||\mu|| \geq ||\mu_2||.$$

Now, let

$$||\beta|| = H$$

$$||y|| = Y$$

$$||\mu|| = U$$

$$||x_1|| = X_1.$$

Therefore we get

$$U \leq \frac{K}{2|\alpha|H} - HY.$$

For the fixed point condition to hold

$$||A(\mu_o,y) - \mu_o|| < (1 - K)||\mu||.$$

Since μ_o has been taken to be zero

$$||A(\mu_o,y)|| < (1 - K)||\mu||$$

$$||-\alpha\beta(\beta y)^2|| < (1 - K)||\mu||$$

or

$$\frac{||-\alpha\beta(\beta y)^2||}{1 - K} < ||\mu||$$

$$\frac{|\alpha|H^3}{1 - K}||y^2|| < U$$

or

$$||y^2|| \leq \frac{1 - K}{|\alpha|H^3} \left[\frac{K}{2|\alpha|H} - HY \right]$$

or

$$\phi_{yy} \leq (1 - K)\left[\frac{K}{2\alpha^2 H^4} - \frac{HY}{|\alpha|H^3} \right]$$

$$= (1 - K)\left[\frac{K - 2|\alpha|H^2 Y}{2\alpha^2 H^4} \right] \tag{3.13}$$

Equations 3.9 and 3.13 give identical constraints on the autocorrelation ϕ_{yy} of the input. However, the first approach is convenient only for the case when the nonlinearity in the feedback loop is of the "square" type and fails to work in the case of "cubic" nonlinearity. In such a case, the second approach can be utilized but it is difficult to get a bound on the input correlation function.

CHAPTER IV

ANALYSIS OF PHASE LOCKED LOOPS

4-1 Introduction

Phase locked loop systems are playing an increasingly important role in modern communication and tracking systems. Phase lock demodulation is widely used for reception of signals in the presence of noise, the major advantage being that demodulation can take place at lower signal power levels than by more conventional means. Recently Kulman and Stratonovich[28] have shown that under certain restrictions, automatic frequency phase control (AFPC) is the optimal device for receiving stochastic signals with fluctuating background noise..

4-2 Physical Behaviour and Description of the Loop

A phase locked loop contains three basic components:

(i) A phase detector (PD).

(ii) A low pass filter.

(iii) A voltage controlled oscillator (VCO), whose frequency is controlled by an external voltage.

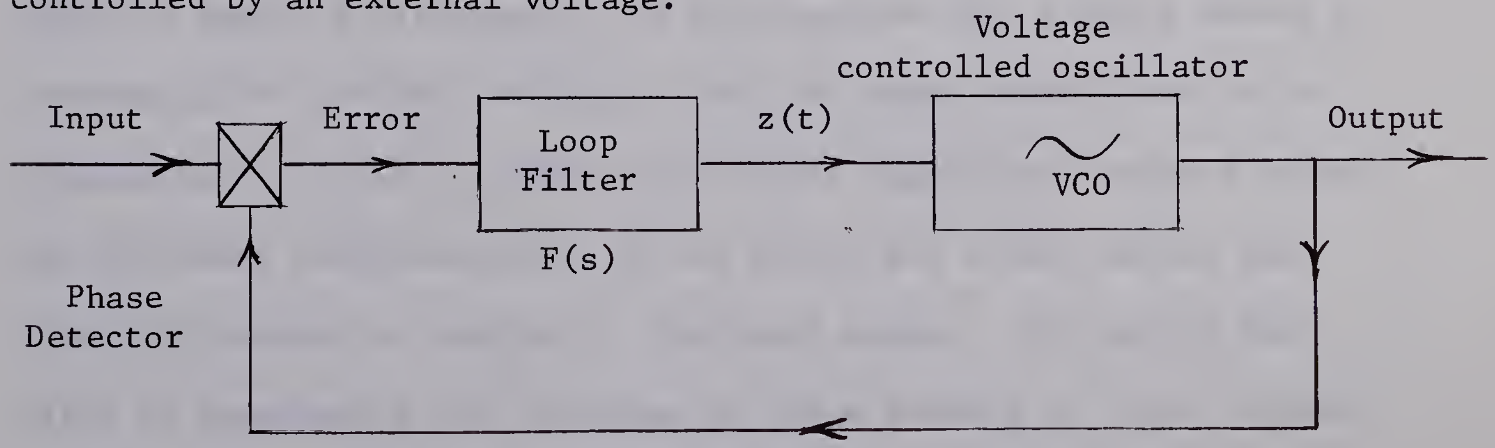


Figure 4.1

The phase detector makes a comparison between the phase of the input and the phase of the VCO. The output of the PD is a function of the phase difference between its two inputs and acts as the error signal in the feedback loop shown in Fig. 4.1. The error signal is filtered by the loop filter and acts as a control signal for the VCO. It tends to cause a frequency change of the VCO so as to reduce the phase difference between the input and the local oscillator. When the loop is synchronized or "locked", the control signal is such that the frequency of the VCO is equal to the average frequency of the input, thereby implying that for each input cycle there exists a unique cycle of the output. This is the basis of application of the phase lock in automatic frequency control[29].

The use of phase lock loops in demodulation can be understood by considering that the input carries the information in its phase or frequency. Invariably additive noises are present. The local oscillator frequency is adjusted close to that expected in the input signal. The output of the phase detector gives instantaneous phase difference. Noise proofing is obtained by averaging the error over a finite length of time. The averaged error controls the frequency of VCO. If the input is stable in frequency, the VCO requires only a small amount of information to perform tracking. Thus the phase locked loop can be considered as a type of filter that passes signals and rejects noise[29]. The pertinent characteristics of the filter are a very narrow band width and automatic tracking of the input signal. The narrow band width is responsible for rejection of large amounts of noise thereby providing a high degree of noise proofing[30].

4-3 Various Mathematical Analytical Approaches

The phase locked loop is a nonlinear system with lag. Several techniques have been evolved for analysing phase locked loop behaviour. Early techniques were based on linear models. Jaffe and Rechtin[30] developed a linearised model for optimum design based on Wiener filtering theory. Develet[31] modified this to a quasi-linear model using the Booton quasi-linearisation model[6]. A correct nonlinear model was developed by Van Trees[32]. The model is a nonlinear, randomly time varying system and is not amenable to analysis. Subsequent work has been based on the simplified model of phase locked loops suggested by Develet[31]. Margolis[33] has given an analysis based on the perturbation method. Van Trees[34] used a Volterra series representation to predict loop performance.

In all the above methods, the design philosophy has basically been oriented to develop a theory which specifies the variance of the phase error as a function of the signal to noise ratio existing in the band width of the loop. The results obtained are approximately correct for large input signal to noise ratios but begin to fail near the threshold.

Another analytic approach used to develop an exact nonlinear theory of phase locked loops is based on the Fokker-Planck method. Viterbi[35] and Tikhonov[36] have given exact results for steady state performance of the first order loops. The central idea underlying the approach is that for systems which are described by first order differential equations whose forcing term is a white Gaussian noise, the output is a simple Markov process. The behaviour of the system can

then be described by either the backward Kolmogorov equation or the Fokker-Planck equation[37]. The main disadvantage of this approach is that analysis can be carried out only for a first order loop without the loop filter included in the feedback circuit. For loops with filters one cannot solve the resulting differential equations. Even for the simplest case, the calculations are tedious and cumbersome.

In the following discussion an anlysis of the first order phase lock loop is presented based on solving the dynamic loop equation using functional techniques, specifically Volterra functional expansion technique. Basically it consists of the application of a perturbation technique[34]. But the analysis differs from Van Trees[34] approach in not using the multidimensional Laplace transform technique and breaking the system into multidimensional kernels. The results obtained give a better estimate of the variance of the phase error near the threshold region for the low signal to noise ratio.

4-4 Simplified Mathematical Model

The model considered in Fig. 4.2 is one suggested by Develet[31].

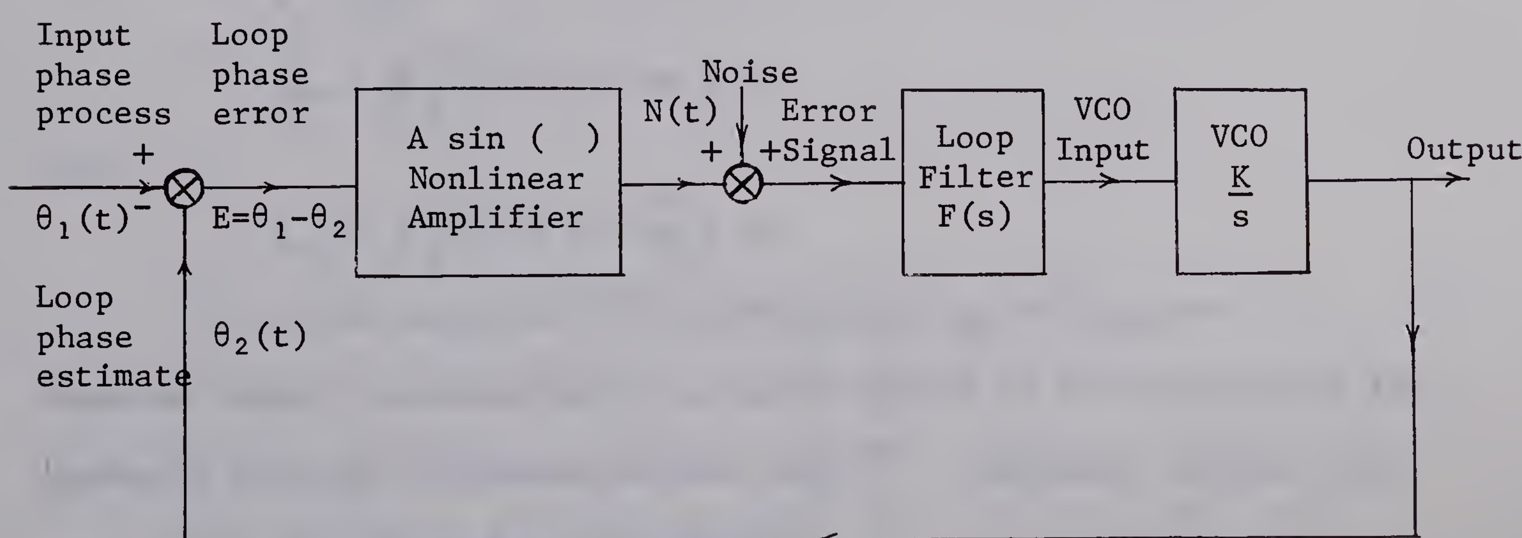


Figure 4.2

Let the input signal in Fig. 4.1 be $\sqrt{2}A\sin[\omega_o(t) + \theta_1]$ where $\theta_1(t)$ can be a random process. Assume that the received noise n(t) is a sample function from a stationary white, narrow band Gaussian process of one-sided spectral density N_o watts/hertz. Assume the phase locked loop is preceeded by a band pass filter with centre frequency f_o and a band width W which is very wide compared to the frequency region of

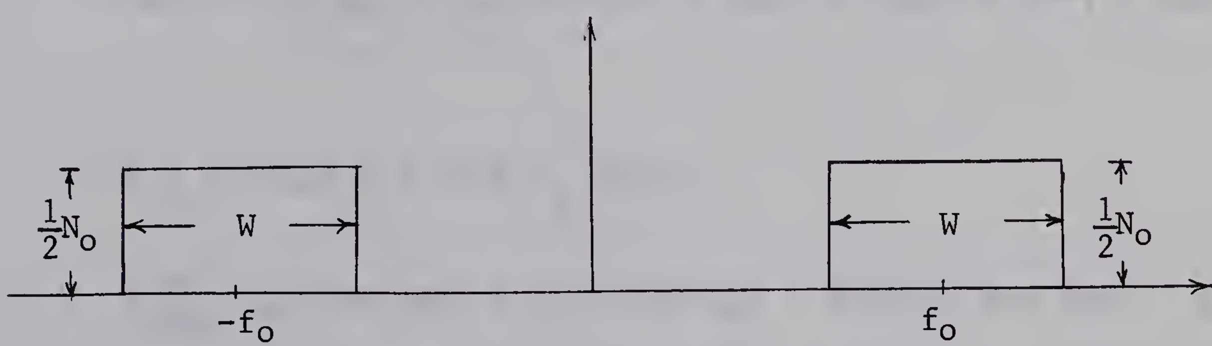


Figure 4.3

interest. $W \le 2f_o$. If $W = 2f_o$, the band pass filter becomes a low pass filter. Expand the given stationary Gaussian process n(t) over an arbitrary interval $0 < t \le T$ by the Fourier series. Thus

$$n(t) = \sum_{m=1}^{\infty} n_{cm}\cos m\omega_1 t + n_{sm}\sin m\omega_1 t$$

where

$$\omega_1 = \frac{2\pi}{T}$$

$$n_{cm} = \frac{2}{T}\int_0^T n(t)\cos m\omega_1 t\, dt$$

and

$$n_{sm} = \frac{2}{T}\int_0^T n(t)\sin m\omega_1 t\, dt.$$

It can be shown that the coefficients n_{cm} and n_{sm} are Gaussian random variables which are uncorrelated as the duration of the expansion interval increases without limit[38]. Introduce the mean frequency of the spectral band by writing

$$m\omega_1 = (m\omega_1 - \omega_o) + \omega_o$$

where

$$\omega_o = 2\pi f_o .$$

Define

$$n_c(t) = \frac{1}{\sqrt{2}} \sum_{m=1}^{\infty} [n_{cm} \cos (m\omega_1 - \omega_o)t + n_{sm} \sin (m\omega_1 - \omega_o)t]$$

and

$$n_s(t) = \frac{1}{\sqrt{2}} \sum_{m=1}^{\infty} [n_{cm} \sin (m\omega_1 - \omega_o)t + n_{sm} \cos (m\omega_1 - \omega_o)t].$$

Then

$$\sqrt{2}\, n_c(t) \cos \omega_o t - \sqrt{2}\, n_s \sin \omega_o t$$

$$= \sum_{m=1}^{\infty} n_{cm}[\cos (m\omega_1 - \omega_o)t \cos \omega_o t - \sin \omega_o t \sin (m\omega_1 - \omega_o)t]$$

$$+ \sum_{m=1}^{\infty} n_{sm}[\sin (m\omega_1 - \omega_o)t \cos \omega_o t + \cos (m\omega_1 - \omega_o)t \sin \omega_o t]$$

$$= \sum_{m=1}^{\infty} n_{cm} \cos m\omega_1 t + \sum_{m=1}^{\infty} n_{sm} \sin m\omega_1 t = n(t).$$

Thus

$$n(t) = \sqrt{2}\, n_c(t) \cos \omega_o t - \sqrt{2}\, n_s(t) \sin \omega_o t.$$

The above decomposition for a narrow band stationary Gaussian process has the following properties[29]:

(i) Spectra of $n_c(t)$ and $n_s(t)$ are low pass in nature.

(ii) $\overline{n(t)} = 0 \Rightarrow \overline{n}_c = \overline{n}_s = 0.$

(iii) $n(t)$ Gaussian $\Rightarrow$ n_c and n_s are also Gaussian.

(iv) $\overline{n^2(t)} = \overline{n_c^2(t)} = \overline{n_s^2(t)}.$

(v) n_c and n_s are independent, that is, $\overline{n_c(t)n_s(t)} = \overline{n_c(t)} \cdot \overline{n_s(t)}.$

Thus $n_c(t)$ and $n_s(t)$ can be regarded essentially as independent white Gaussian processes of one sided spectral density N watts/hertz.

The output of the Voltage Controlled Oscillator (VCO) in

Fig. 4.1 is a sinusoid whose frequency is controlled by the input voltage z(t),

$$\dot{\theta}_2(t) = \frac{d\theta_2(t)}{dt} = K_1 z(t)$$

so that when $z(t) = 0$, the VCO frequency is ω_o. Hence the VCO output is

$$\sqrt{2}\, K_2 \cos\,[\omega_o t + \theta_2(t)].$$

Assume that the phase detector is a multiplier. Therefore, the product of the input and the reference signal is

$$2\{A \sin\,[\omega_o(t) + \theta_1(t)] + n_c(t) \cos \omega_o t - n_s(t) \sin \omega_o t\}$$

$$\cdot\ \{K_2 \cos\,[\omega_o t + \theta_2(t)]\}$$

$$= K_2\{A \sin\,[\theta_1(t) - \theta_2(t)] + n_c(t) \cos \theta_2(t) + n_s(t) \sin \theta_2(t)$$

$$+ A \sin\,[2\omega_o t + \theta_1(t) + \theta_2(t)] + n_c(t) \cos\,[2\omega_o t + \theta_2(t)]$$

$$- n_s(t) \sin\,[2\omega_o t + \theta_2(t)]\}.$$

The double frequency terms can be neglected since neither the filter nor the VCO would respond to these for large ω_o. Therefore,

$$z(t) = K_2 F(s)\{A \sin\,[\theta_1(t) - \theta_2(t)] + n_c(t) \cos \theta_2(t)$$

$$+ n_s(t) \sin \theta_2(t)\}$$

where F(s) is a rational function which represents, in operational notation, the effect of a linear filter in the loop[35].

Let $E(t) = \theta_1(t) - \theta_2(t)$ denote the instantaneous phase error. Therefore

$$\dot{E}(t) = \dot{\theta}_1(t) - K_1 z(t)$$

$$= \dot{\theta}_1(t) - K_1 K_2 F(s)\,[A \sin E(t) + N(t)]$$

where

$$N(t) = n_c(t) \cos \theta_2(t) + n_s(t) \sin \theta_2(t).$$

Let $K_1K_2 = K$. Then

$$E(t) = \theta_1(t) - [A \sin E(t) + N(t)]F(s)\frac{K}{s}. \qquad 4.1$$

The differential equation 4.1 in operational form represents the dynamic operation of the phase locked loop with the block diagram given in Fig. 4.2. Viterbi[35] has shown that N(t) is a stationary process with exactly the same statistics as $n_c(t)$ and $n_s(t)$. Hence N(t) is a Gaussian white process at least over the frequency range up to ω_o rad/sec with one sided spectral density N_o watts/hertz.

4-5 Analysis of First Order Loop

When considering the first order loop, F(s) = 1 and the phase locked loop model reduces to Fig. 4.4.

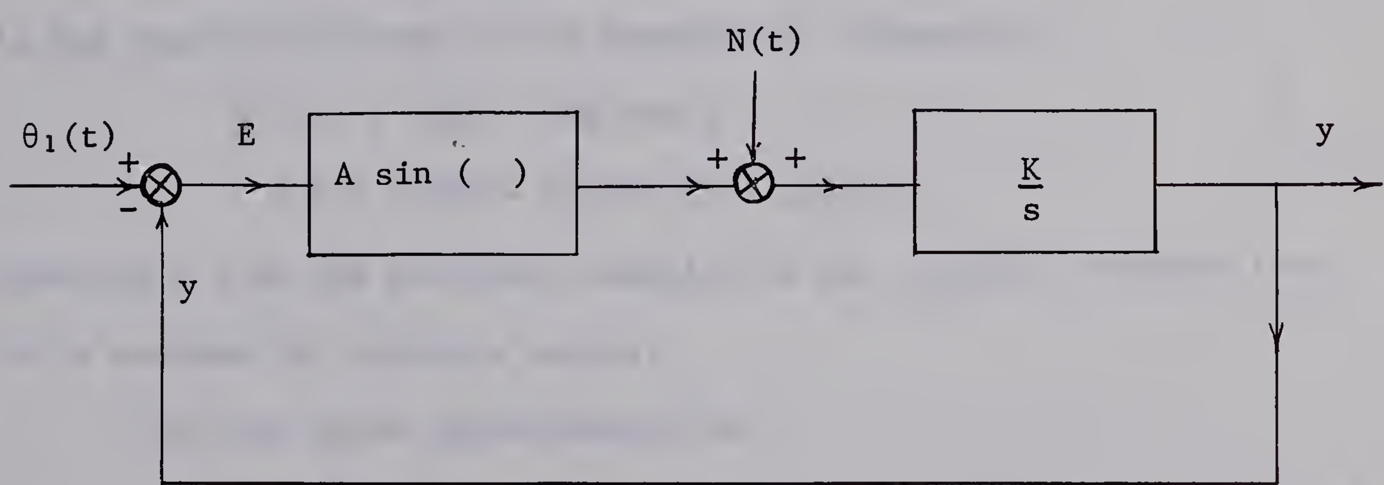


Figure 4.4

Equation 4.1 becomes

$$\frac{dE}{dt} + KA \sin E(t) = \dot{\theta}_1(t) - KN(t)$$

$$= x(t) \qquad 4.2$$

where x(t) can be considered as an equivalent input. Assume $\dot{\theta}_1(t) = 0$. N(t) is a sample function from a white Gaussian process with zero mean

and correlation function

$$R_N(\tau) = \overline{N(t)N(t - \tau)}$$
$$= \frac{N_o}{2}\,\delta[t - (t - \tau)]$$
$$= \frac{N_o}{2}\,\delta(\tau).$$

Equation 4.2 can be written as

$$\dot{E} + KAE + KA\sin E - KAE = x(t)$$

or

$$L(E) + KA\sin E - KAE = x(t)$$

where L is the linear differential operator (s + KA).

Let β be the corresponding integral operator, that is

$$\beta(x) = \int_0^\infty h(t - \tau)x(\tau)d\tau$$

and

$$h(t) = \mathcal{L}^{-1}\frac{1}{s + KA} = e^{-KAt}$$

is the impulse response of the operator β. Therefore

$$\left.\begin{aligned} E &= \beta x + \beta KAE - \beta KA\sin E \\ &= \beta(x + KAE - KA\sin E) = \mu(E,x) \end{aligned}\right] \quad 4.3$$

Equation 4.3 is the nonlinear equation in the integral operator form to be analysed by Volterra series.

Let the first approximation be

$$E_o = \beta x. \qquad 4.4$$

Second approximation

$$E_1 = \beta(x + KAE_o - KA\sin E_o) \qquad 4.5$$

etc.

The solution of Equation 4.3, E* can be written as

$$E^* = E_o + (E_1 - E_o) + \ldots + (E_n - E_{n-1}) + \ldots \qquad 4.6$$

provided the series 4.6 converges. In Equation 4.6, E* is obtained

by Volterra series generated through an iteration proceedure. A convergence criterion is now formulated for the Volterra series so obtained, based on the contraction mapping principle. This approach has been discussed in Chapter III.

Consider the nonlinear operator Equation 4.3

$$E = \beta[x + KAE - KA \sin E]$$
$$= \mu[E,x].$$

For μ to be a contraction operator,

$$||\mu(E_2,x) - \mu(E_1,x)|| \leq Y||E_2 - E_1||$$

where Y is a constant, $0 \leq Y < 1$.

i.e.

$$||\beta(x + KAE_2 - KA \sin E_2) - \beta(x + KAE_1 - KA \sin E_1)||$$
$$\leq Y||E_2 - E_1||.$$

i.e.

$$KA||\beta|| \; ||E_2 - \sin E_2 - E_1 + \sin E_1|| \leq Y||E_2 - E_1||.$$

Expanding $\sin E_2$ and $\sin E_1$ in power series, we get

$$E_2 - E_1 + \sin E_1 - \sin E_2 = \left[-\frac{E_1^{\,3}}{3!} + \frac{E_1^{\,5}}{5!} - \frac{E_1^{\,7}}{7!} + \ldots\right]$$
$$- \left[-\frac{E_2^{\,3}}{3!} + \frac{E_1^{\,5}}{5!} - \frac{E_1^{\,7}}{7!} + \ldots\right].$$

Since $E_2 - E_1$ is a common factor to all the terms, we get

$$E_2 - E_1 + \sin E_1 - \sin E_2 = (E_2 - E_1)\left[\frac{E_2^{\,2} + E_1E_2 + E_1^{\,2}}{3!}\right.$$
$$- \frac{E_2^{\,4} + E_2^{\,3}E_1 + E_2^{\,2}E_1^{\,2} + E_2E_1^{\,3} + E_1^{\,4}}{5!}$$
$$\left. + \cdots\right].$$

Hence

$$||E_2 - E_1 + \sin E_1 - \sin E_2|| \leq ||E_2 - E_1||\left[\frac{3||E||^2}{3!} + \frac{5||E||^4}{5!}\right.$$

$$+ \frac{7||E||^6}{7!} + \cdots].$$

$$= ||E_2 - E_1||[\frac{||E||^2}{2!} + \frac{||E||^4}{4!}$$

$$+ \frac{||E||^6}{6!} + \cdots]$$

$$= ||E_2 - E_1||[\cosh ||E|| - 1].$$

Therefore the contraction condition is satisfied if

$$KA||\beta||[\cosh ||E|| - 1]||E_2 - E_1|| \leq ||E_2 - E_1||Y$$

or

$$\cosh ||E|| \leq 1 + \frac{Y}{KA||\beta||}$$

or

$$||E|| \leq \cosh^{-1}[1 + \frac{Y}{KA||\beta||}].$$

For the fixed point condition to be satisified

$$||\mu(E_o) - E_o|| < (1 - Y)(||E||).$$

We have

$$E_o = \beta x.$$

Therefore

$$\mu(E_o) = \beta[x + KAE_o - KA \sin E_o]$$

$$= \beta[x + KA\beta x - KA \sin \beta x]$$

and

$$\mu(E_o) - E_o = KA\beta(\beta x - \sin \beta x).$$

Hence

$$||KA\beta(\beta x - \sin \beta x)|| < (1 - Y)||E||$$

or

$$KA||\beta||[||\beta x|| + ||\sin \beta x||] < (1 - Y)||E||.$$

But

$$||\sin y|| \le ||y||.$$

Therefore for the fixed point condition to be satisfied

$$KA||\beta|| \; 2||\beta x|| < (1 - Y)||E||$$

or

$$||x|| \le \frac{1 - Y}{2KAH^2} \cosh^{-1}\left[1 + \frac{Y}{KAH}\right] \qquad 4.7$$

where

$$||\beta|| = H.$$

Now consider the second approximation

$$E_1 = \beta(x + KAE_o - KA \sin E_o). \qquad 4.5$$

The object of interest is $< E_1{}^2 >$

$$\sin E_o = E_o - \frac{E_o{}^3}{3!} + \frac{E_o{}^5}{5!} + \cdots .$$

Substituting this value in Equation 4.5

$$E_1 = \beta[x + KAE_o - KAE_o + KA\frac{E_o{}^3}{3!} - KA\frac{E_o{}^5}{5!} + \cdots].$$

Considering 5^{th} power terms in E_o only

$$E_1 = \beta[x + KA\frac{E_o{}^3}{3!} - KA\frac{E_o{}^5}{5!}]$$

$$= \beta x + KA\beta\frac{E_o{}^3}{3!} - KA\beta\frac{E_o{}^5}{5!}$$

$$= E_o + KA\beta\frac{E_o{}^3}{3!} - KA\beta\frac{E_o{}^5}{5!}.$$

Therefore

$$< E_1{}^2 > = < E_o{}^2 > + < 2 \cdot \frac{KA}{3!}(\beta E_o{}^3)(E_o) > +$$

$$< -\frac{2KA}{5!}(E_o)(E_o) > + < -\frac{2K^2A^2}{3!5!}(\beta E_o{}^3)(\beta E_o{}^5) >$$

$$+ < \frac{K^2A^2}{3!3!}\beta^2(E_o)^6 > + < \frac{K^2A^2}{5!5!}\beta^2(E_o)^{10} >$$

$$= (a) + (b) + (c) + (d) + (e) + (f). \qquad 4.8$$

Consider the expression 4.8a viz. $< E_o{}^2 >$

$$E_o(t) = \beta x$$

$$= -\beta KN(t)$$

$$= - \int_0^\infty h(\tau)KN(t - \tau)d\tau$$

where

$$h(\tau) = e^{-KA\tau}, \qquad \tau \geq 0$$

$$= 0, \qquad \tau < 0.$$

Therefore

$$<E_o{}^2(t)> = K^2 \int_0^\infty \int_0^\infty e^{-KA\tau_1} \cdot e^{-KA\tau_2} <N(t - \tau_1)N(t - \tau_2)> d\tau_1 d\tau_2$$

$$<N(t - \tau_1)N(t - \tau_2)> = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} N(t - \tau_1)N(t - \tau_2)dt$$

put

$$t - \tau_1 = \gamma$$

then

$$dt = -d\gamma$$

and the limits of integration are

$$\gamma = T - \tau_1$$

and

$$\gamma = -T - \tau_1.$$

Therefore

$$<N(t - \tau_1)N(t - \tau_2)> = \lim_{T\to\infty} \frac{1}{2T}\int_{-T-\tau_1}^{T-\tau_1} N(\gamma)N(\gamma + \tau_1 - \tau_2)d\gamma.$$

The correlation function of N(t) is defined by

$$R_N(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} N(t)N(t + \tau)dt$$

$$= \frac{N_o}{2}\delta(\tau).$$

Therefore

$$\langle N(t - \tau_1)N(t - \tau_2)\rangle = \frac{N_o}{2}\delta(\tau_1 - \tau_2)$$

and

$$\langle E_o{}^2\rangle = K^2\int_0^\infty\int_0^\infty e^{-KA\tau_1}\cdot e^{-KA\tau_2}\frac{N_o}{2}\delta(\tau_1 - \tau_2)d\tau_1 d\tau_2$$

$$= \frac{K^2N_o}{2}\int_0^\infty d\tau_1 e^{-KA\tau_1}\int_0^\infty e^{-KA\tau_2}\delta(\tau_1 - \tau_2)d\tau_2$$

$$= \frac{K^2N_o}{2}\int_0^\infty d\tau_1 e^{-KA\tau_1}\cdot e^{-KA\tau_1}$$

$$= \frac{K^2N_o}{2}\left[\frac{e^{-2KA\tau_1}}{-2KA}\right]_0^\infty$$

$$= \frac{KN_o}{4A}$$

$$\doteq Z \qquad 4.9$$

The parameter Z plays an important role

$$Z = \frac{N_o\left(\frac{AK}{4}\right)}{(A)^2}$$

In the linearised model of Fig. 4.4, the sinusoidal nonlinearity is replaced by its gain A about $E = 0$ and $\langle E^2\rangle = Z$,[35] thereby implying the variance of E is the same as the noise power at the output of an ideal low pass filter of bandwidth $\frac{KA}{4}$ when the input is white noise of one sided spectral density N_o. Hence for the first order loop, loop bandwidth $B_L = \frac{AK}{4}$ and $Z = \frac{N_oB_L}{(A)^2}$. But A^2 is the received signal power. Therefore $\frac{1}{Z} = \frac{A^2}{N_oB_L}$ is the signal to noise ratio (SNR) in the bandwidth of the loop[29].

Now consider expression 4.8b viz.

$$\langle\frac{2}{3!}KA(\beta E_o)^3E_o\rangle$$

$$\beta(E_o)^3 = \int_0^\infty h(\tau_1)E_o{}^3(t - \tau_1)d\tau_1$$

[illegible]

[illegible]

$$[illegible]$$

[illegible]

[illegible]

$$[illegible]$$

where

$$E_o(t) = \int_0^\infty h(\tau)x(t - \tau)d\tau.$$

Hence

$$\beta(E_o)^3 = \int_0^\infty d\tau \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_0^\infty d\tau_3 h(\tau)h(\tau_1)h(\tau_2)h(\tau_3) \cdot$$
$$x(t - \tau - \tau_1)x(t - \tau - \tau_2)x(t - \tau - \tau_3).$$

Therefore

$$\langle\beta(E_o)^3 E_o\rangle = \langle\beta(E_o)^3 \beta x\rangle$$
$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty d\tau d\tau_1 d\tau_2 d\tau_3 d\tau_4 h(\tau)h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4) \cdot$$
$$\langle x(t - \tau_4)x(t - \tau - \tau_1)x(t - \tau - \tau_2)x(t - \tau - \tau_3)\rangle \tag{4.10}$$

Consider n normal real random variables $x_1, x_2, \ldots, x_n$. Let $z_j = x_j - \bar{x}_j$, where $\bar{x}_j$ = mean value of x. Then it can be shown[39] that

$$E\{z_1, \ldots, z_{2m}\} = \sum_{\text{all pairs}} \left(\prod_{j \neq k}^{m} z_j z_k\right)$$
$$= \sum_{\text{all pairs}} \{\langle z_j z_k\rangle\langle z_1 z_p\rangle \ldots \langle z_q z_s\rangle\} \quad j \neq k,\ 1 \neq p, \text{ etc.}$$

and

$$E\{z_1, z_2, \ldots, z_{2m+1}\} = 0.$$

Now the number of such averages over pairs is equal to the number of different ways 2m different variables $z_1, \ldots, z_{2m}$ can be chosen in pairs, which is[39] $\frac{2m!}{2^m \cdot m!}$.

In Equation 4.10 there are four random variables so that m = 2. Hence there are $\frac{4!}{2^2 \cdot 2!} = 3$ ways into which

$$\langle x(t - \tau_4)x(t - \tau - \tau_1)x(t - \tau - \tau_2)x(t - \tau - \tau_3)\rangle$$

factors into the product of two terms taken at a time.

Therefore

$$<x(t - \tau_4)x(t - \tau - \tau_1)x(t - \tau - \tau_2)x(t - \tau - \tau_3)>$$

$$= <x(t - \tau_4)x(t - \tau - \tau_1)><x(t - \tau - \tau_2)x(t - \tau - \tau_3)>$$

$$+ <x(t - \tau - \tau_1)x(t - \tau - \tau_2)><x(t - \tau - \tau_3)x(t - \tau_4)$$

$$+ <x(t - \tau_4)x(t - \tau - \tau_2)><x(t - \tau - \tau_1)x(t - \tau - \tau_3)>$$

$$= K^4(\frac{N_o}{2})^2\Big[\delta(\tau_4 - \tau - \tau_1)\delta(\tau + \tau_2 - \tau - \tau_3) +$$

$$+ \delta(\tau_1 - \tau_2)\delta(\tau + \tau_3 - \tau_4) + \delta(\tau_1 - \tau_3)\ (\tau + \tau_2 - \tau_4)\Big] \qquad 4.11$$

Let us evaluate the contribution of the first term of Equation 4.11 to Equation 4.10.

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty d\tau d\tau_1 \ldots d\tau_4 h(\tau)h(\tau_1)\ldots h(\tau_4)\delta(\tau_4 - \tau - \tau_1)\delta(\tau_2 - \tau_3) =$$

$$\int_0^\infty\int_0^\infty\int_0^\infty h(\tau_4 - \tau_1)h(\tau_1)h(\tau_2)h(\tau_2)h(\tau_4)d\tau_1 d\tau_2 d\tau_4. \qquad 4.12$$

Similarly

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty d\tau d\tau_1 \ldots d\tau_4 h(\tau)h(\tau_1)\ldots h(\tau_4)\delta(\tau_1 - \tau_2)\delta(\tau + \tau_3 - \tau_4)$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty h(\tau_3 - \tau_4)h(\tau_2)h(\tau_2)h(\tau_3)h(\tau_4)d\tau_2 d\tau_3 d\tau_4$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty h(\tau_3 - \tau_1)h(\tau_2)h(\tau_2)h(\tau_3)h(\tau_1)d\tau_2 d\tau_3 d\tau_1$$

(because τ_1 and τ_4 are symmetric)

$$= \int_0^\infty\int_0^\infty\int_0^\infty h(\tau_4 - \tau_1)h(\tau_1)h(\tau_2)h(\tau_2)h(\tau_4)d\tau_1 d\tau_2 d\tau_4. \qquad 4.13$$

(interchanging τ_4 and τ_3 because of symmetry)

Equations 4.12 and 4.13 are identical. Similarly by the use of symmetry of arguments the contribution of the third term of Equation 4.11 to Equation 4.10 is identical to the other two terms. Therefore

$$<\beta(E_o)^3 E_o> = 3K^4(\frac{No}{2})^2\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty h(\tau)h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)\delta(\tau_4 - \tau - \tau_1)$$

$$\delta(\tau_2 - \tau_3) \cdot d\tau d\tau_1 d\tau_2 d\tau_3 d\tau_4$$

$$= 3K^4\left(\frac{N_o}{2}\right)^2\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty h(\tau)h(\tau_1)h(\tau_3)h(\tau_3)h(\tau_4)\delta(\tau_4 - \tau - \tau_1)d\tau d\tau_1 \ldots$$

$$d\tau_4$$

$$= 3K^4\left(\frac{N_o}{2}\right)^2\int_0^\infty\int_0^\infty\int_0^\infty h(\tau)h(\tau_1)h^2(\tau_3)h(\tau + \tau_1)d\tau d\tau_1 d\tau_3$$

$$= 3K^4\left(\frac{N_o}{2}\right)^2\int_0^\infty\int_0^\infty h(\tau)h(\tau_1)h(\tau + \tau_1)\int_0^\infty e^{-2KA\tau_3}d\tau_3$$

$$= 3K^4\left(\frac{N_o}{2}\right)^2\frac{1}{2KA}\int_0^\infty\int_0^\infty e^{-2KA(\tau + \tau_1)}d\tau d\tau_1$$

$$= 3K^4\left(\frac{N_o}{2}\right)^2\left(\frac{1}{2KA}\right)^3.$$

Hence

$$< \frac{2}{3!}KA(\beta E_o)^3 E_o > = \frac{2}{3\cdot 2}\cdot 3K^5\left(\frac{N_o}{2A}\right)^2\left(\frac{1}{2K}\right)^3$$

$$= \frac{1}{2}\left(\frac{KN_o}{4A}\right)^2$$

$$= \frac{1}{2}Z^2. \qquad 4.14$$

Now consider expression 4.8c, $< -\frac{2K}{5!}(\beta E_o)^5(E_o) >$

$$\beta(E_o)^5 = \int_0^\infty h(\tau_1)E_o{}^5(t - \tau_1)d\tau_1$$

where

$$E_o(t) = \int_0^\infty h(\tau)x(t - \tau)d\tau.$$

Therefore

$$\beta(E_o)^5 = \int_0^\infty d\tau_1\ldots\int_0^\infty d\tau_6 h(\tau_1)\ldots h(\tau_6)x(t - \tau_1 - \tau_2)x(t - \tau_1 - \tau_3)\cdot$$
$$x(t - \tau_1 - \tau_4)x(t - \tau_1 - \tau_5)x(t - \tau_1 - \tau_6).$$

Therefore

$$<\beta(E_o)^5 E_o> = \int_0^\infty\ldots\int_0^\infty d\tau d\tau_1\ldots d\tau_6 h(\tau)h(\tau_1)\ldots h(\tau_6)\cdot$$
$$<x(t - \tau)x(t - \tau_1 - \tau_2)x(t - \tau_1 - \tau_3)$$
$$x(t - \tau_1 - \tau_4)\cdot x(t - \tau_1 - \tau_5)x(t - \tau_1 - \tau_6)>. \qquad 4.17$$

In Equation 4.17, there are six Gaussian random variables. Therefore

the number of factors is $\frac{6!}{2^3 \cdot 3!} = 15$. Making use of the symmetry of arguments as done while evaluating the expression 4.8b, one obtains that all the 15 terms are the same[34]. Hence

$$\begin{aligned}
<\beta(E_o)^5 E_o> &= \int_0^\infty \ldots \int_0^\infty d\tau d\tau_1 \ldots d\tau_6 h(\tau)h(\tau_1)\ldots h(\tau_6) \times 15\{<x(t-\tau) \\
&\quad x(t-\tau_1-\tau_2)><x(t-\tau_1-\tau_3)x(t-\tau_1-\tau_4)> \cdot \\
&\quad <x(t-\tau_1-\tau_5)x(t-\tau_1-\tau_6)>\} \\
&= 15K^6(\tfrac{N_o}{2})^3 \int_0^\infty \ldots \int_0^\infty d\tau d\tau_1 \ldots d\tau_6 h(\tau)h(\tau_1)\ldots h(\tau_6) \cdot \delta(\tau-\tau_1-\tau_2) \\
&\quad \delta(\tau_3-\tau_4)\ (\tau_5-\tau_6) \\
&= 15K^6(\tfrac{N_o}{2})^3 \int_0^\infty \ldots \int_0^\infty h(\tau_1+\tau_2)h(\tau_4)h(\tau_6) \cdot h(\tau_1)h(\tau_2)h(\tau_4)h(\tau_6) \\
&\quad d\tau_1 d\tau_2 d\tau_4 d\tau_6 \\
&= 15K^6(\tfrac{N_o}{2})^3 \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-2KA(\tau_1+\tau_2+\tau_4+\tau_6)} d\tau_1 d\tau_2 d\tau_4 d\tau_6 \\
&= 15K^6(\tfrac{N_o}{2})^3 (\tfrac{1}{2KA})^4
\end{aligned}$$

Therefore

$$\begin{aligned}
<-\frac{2KA}{5!}(\beta E_o{}^5)E_o> &= -\frac{2}{5\cdot4\cdot3\cdot2} KA \cdot 15K^6(\frac{N_o}{2})^3 \frac{1}{16K^4A^4} \\
&= -\frac{1}{8}(\frac{KN_o}{4A})^3 \\
&= -\frac{1}{8}Z^3. \qquad 4.18
\end{aligned}$$

Now consider expression 4.8d, viz. $<\frac{K^2A^2}{3!3!}\beta^2 E_o{}^6>$

$$\begin{aligned}
<\beta^2 E_o{}^6> &= <\beta E_o{}^3 \cdot \beta E_o{}^3> \\
&= \int_0^\infty \ldots \int_0^\infty d\tau_1 d\tau_2 \ldots d\tau_6 d\tau_\alpha d\tau_\beta h(\tau_1)\ldots h(\tau_6) \cdot \\
&\quad h(\tau_\alpha)h(\tau_\beta) \cdot <x(t-\tau_\alpha-\tau_1)x(t-\tau_\alpha-\tau_2) \\
&\quad x(t-\tau_\alpha-\tau_3) \cdot x(t-\tau_\beta-\tau_4)x(t-\tau_\beta-\tau_5) \\
&\quad x(t-\tau_\beta-\tau_6)>. \qquad 4.19
\end{aligned}$$

Sixth order moments factors giving 15 terms.

Utilizing the symmetry of argument and the properties of δ function one gets[34]

$$<x(t - \tau_\alpha - \tau_1)x(t - \tau_\alpha - \tau_2)x(t - \tau_\alpha - \tau_3) \cdot x(t - \tau_\beta - \tau_4)x(t - \tau_\beta - \tau_5)$$
$$x(t - \tau_\beta - \tau_6) =$$
$$K^6\left(\frac{N_o}{2}\right)^3\Big[9\delta(\tau_1 - \tau_2)\delta(\tau_\alpha + \tau_3 - \tau_\beta - \tau_4)\delta(\tau_5 - \tau_6) + 6\delta(\tau_\alpha + \tau_1 - \tau_\beta - \tau_4)$$
$$\delta(\tau_\alpha + \tau_2 - \tau_\beta - \tau_5) \cdot \delta(\tau_\alpha + \tau_3 - \tau_\beta - \tau_6)\Big]. \quad 4.20$$

Therefore

$$<\beta^2 E_o{}^6> = K^6\left(\frac{N_o}{2}\right)^3\left[\begin{array}{l}\int_0^\infty\int_0^\infty\int_0^\infty\cdots\int_0^\infty d\tau_1\ldots d\tau_6 d\tau_\alpha d\tau_\beta h(\tau_1)h(\tau_2)\ldots h(\tau_6)h(\tau_\alpha) \\ h(\tau_\beta) \cdot \{9\delta(\tau_1 - \tau_2)\delta(\tau_\alpha + \tau_3 - \tau_\beta - \tau_4) \cdot \\ \delta(\tau_5 - \tau_6) + 6\delta(\tau_\alpha + \tau_1 - \tau_\beta - \tau_4) \cdot \\ \delta(\tau_\alpha + \tau_2 - \tau_\beta - \tau_5) \cdot \delta(\tau_\alpha + \tau_3 - \tau_\beta - \tau_6)\}\end{array}\right]. \quad 4.21$$

$$= K^6\left(\frac{N_o}{2}\right)^3\left[\begin{array}{l}9\{\left(\int_0^\infty h^2(\tau_1)d\tau_1\right)^2\int_0^\infty h(\tau_3)d\tau_3\int_0^\infty h(\tau_\alpha)d\tau_\alpha\int_0^\infty h(\tau_\beta)d\tau_\beta \\ h(\tau_\alpha + \tau_3 - \tau_\beta)u(\tau_\alpha + \tau_3 - \tau_\beta)\} + \\ 6\{\int_0^\infty h(\tau_\alpha)d\tau_\alpha\int_0^\infty h(\tau_\beta)d\tau_\beta\int_0^\infty h(\tau_1)d\tau_1\int_0^\infty h(\tau_2)d\tau_2\int_0^\infty h(\tau_3)d\tau_3 \cdot \\ h(\tau_\alpha - \tau_\beta + \tau_1)u(\tau_\alpha - \tau_\beta + \tau_1) \cdot h(\tau_\alpha - \tau_\beta + \tau_2) \cdot \\ u(\tau_\alpha - \tau_\beta + \tau_2) \cdot h(\tau_\alpha - \tau_\beta + \tau_3)u(\tau_\alpha - \tau_\beta + \tau_3)\}\end{array}\right]. \quad 4.22$$

$$\left[\int_0^\infty h^2(\tau_1)d\tau_1\right]^2 = \left[\int_0^\infty e^{-2KA\tau_1}d\tau_1\right]^2$$
$$= \frac{1}{4K^2A^2}$$

Consider

$$\int_0^\infty\int_0^\infty\int_0^\infty h(\tau_\alpha)h(\tau_\beta)h(\tau_3)h(\tau_\alpha + \tau_3 - \tau_\beta)u(\tau_\alpha + \tau_3 - \tau_\beta)d\tau_\alpha d\tau_\beta d\tau_3 =$$
$$\int_0^\infty d\tau_\alpha h(\tau_\alpha)\int_0^\infty d\tau_3 h(\tau_3)\int_0^\infty h(\tau_\beta)h(\tau_\alpha + \tau_3 - \tau_\beta)u(\tau_\alpha + \tau_3 - \tau_\beta)d\tau_\beta. \quad 4.23$$

Let $\tau_\alpha + \tau_3 - \tau_\beta = \xi$

Then

$$\int_0^\infty h(\tau_\beta)h(\tau_\alpha + \tau_3 - \tau_\beta)u(\tau_\alpha + \tau_3 - \tau_\beta)d\tau_\beta = -\int_{\tau_\alpha+\tau_3}^{\infty} h(\xi)h(\tau_\alpha + \tau_3 - \xi) \cdot u(\xi)d(\xi)$$

$$= \int_{-\infty}^{\tau_\alpha+\tau_3} h(\xi)h(\tau_\alpha + \tau_3 - \xi) \cdot u(\xi)d\xi. \qquad 4.24$$

Whenever $\xi < 0$, the integrand in 4.24 is zero. Hence

$$\int_0^\infty h(\tau_\beta)h(\tau_\alpha + \tau_3 - \tau_\beta)u(\tau_\alpha + \tau_3 - \tau_\beta)d\tau_\beta = \int_0^{\tau_\alpha+\tau_3} h(\xi)h(\tau_\alpha + \tau_3 - \xi)d\xi$$

$$= \int_0^{\tau_\alpha+\tau_3} e^{-KA\xi}e^{-KA(\tau_\alpha+\tau_3-\xi)}d\xi$$

$$= e^{-KA\tau_\alpha-KA\tau_3}(\tau_\alpha + \tau_3).$$

Hence the first term in [ ] bracket in Equation 4.22 reduces to

$$\frac{9}{4K^2A^2} \cdot 2\int_0^\infty e^{-2KA\tau_\alpha}d\tau_\alpha\int_0^\infty \tau_3 e^{-2KA\tau_3}d\tau_3 = \frac{9}{4K^2A^2} \cdot \frac{2}{2KA}\Big[\tau_3(-\frac{1}{2KA}e^{-2KA\tau_3}$$

$$+ \int\frac{1}{2KA}e^{-2KA\tau_3} \cdot 1d\tau_3\Big]_0^\infty$$

$$= \frac{9}{4K^3A^3}\Big[\frac{-\tau_3}{2KAe^{+2KA\tau_3}} - \frac{1}{4K^2A^2}e^{-2KA\tau_3}\Big]_0^\infty$$

$$= \frac{9}{4K^3A^3} \cdot \frac{1}{4K^2A^2}$$

$$= \frac{9}{16K^5A^5}. \qquad 4.25$$

Now consider

$$\int_0^\infty\cdots\int_0^\infty d\tau_1 d\tau_2\ldots d\tau_6 d\tau_\alpha d\tau_\beta h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)h(\tau_5)h(\tau_6)h(\tau_\alpha)h(\tau_\beta) \cdot$$

$$\delta(\tau_\alpha + \tau_1 - \tau_\beta - \tau_4)\delta(\tau_\alpha + \tau_2 - \tau_\beta - \tau_5)\delta(\tau_\alpha + \tau_3 - \tau_\beta - \tau_6)$$

$$= \int_0^\infty e^{-KA\tau_\alpha}d\tau_\alpha\int_0^\infty e^{-KA\tau_\beta}d\tau_\beta\int_0^\infty e^{-KA\tau_1}d\tau_1\int_0^\infty e^{-KA\tau_2}d\tau_2\int_0^\infty e^{-KA\tau_3}d\tau_3 \cdot e^{-KA(\tau_\alpha-\tau_\beta+\tau_1)}$$

$$u(\tau_\alpha - \tau_\beta + \tau_1) \cdot e^{-KA(\tau_\alpha-\tau_\beta+\tau_2)}u(\tau_\alpha - \tau_\beta + \tau_2) \cdot e^{-KA(\tau_\alpha-\tau_\beta+\tau_3)}$$

$$u(\tau_\alpha - \tau_\beta + \tau_3)$$

$$= \int_0^\infty e^{-4KA\tau_\alpha}d\tau_\alpha\int_0^\infty e^{+2KA\tau_\beta}d\tau_\beta\int_0^\infty e^{-2KA\tau_2}u(\tau_\alpha - \tau_\beta + \tau_2)d\tau_2 \cdot \int_0^\infty e^{-2KA\tau_3}$$

$$u(\tau_\alpha - \tau_\beta + \tau_3)d\tau_2\int_0^\infty e^{-2KA\tau_1}u(\tau_\alpha - \tau_\beta + \tau_1)d\tau_1 =$$

$$\int_0^\infty e^{-4KA\tau_\alpha} d\tau_\alpha \int_0^\infty e^{+2KA\tau_\beta} d\tau_\beta \left[\int_0^\infty e^{-2KA\tau} u(\tau_\alpha - \tau_\beta + \tau) d\tau\right]^3. \qquad 4.26$$

Consider the term $[\int(\ldots)d\tau]^3$ in Equation 4.26. For $\tau + \tau_\alpha - \tau_\beta < 0$ the integrand is zero. Hence $\tau > \tau_\beta - \tau_\alpha$ for the integral to exist. Also $\tau, \tau_\alpha, \tau_\beta \geq 0$. There are two situations to consider.

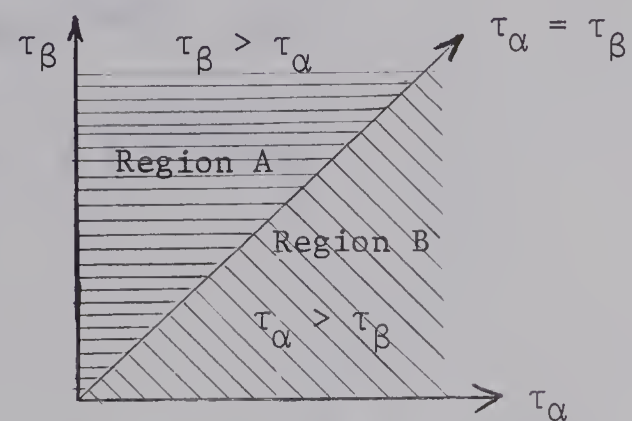


Figure 4.5

(i) $\tau_\beta > \tau_\alpha$. This corresponds to region A. In this case, the limits of integration are

$$\tau_\beta - \tau_\alpha < \tau < \infty$$

$$0 < \tau_\alpha < \tau_\beta$$

$$0 < \tau_\beta < \infty.$$

(ii) $\tau_\alpha > \tau_\beta$. This corresponds to region B and τ is always greater than $\tau_\beta - \tau_\alpha$. Therefore the limits of integration are

$$0 < \tau < \infty$$

$$0 < \tau_\alpha < \infty$$

$$0 < \tau_\beta < \tau_\alpha.$$

Hence the integral in Equation 4.26 reduces to

$$\int_0^\infty e^{+2KA\tau_\beta} d\tau_\beta \int_0^{\tau_\beta} e^{-4KA\tau_\alpha} d\tau_\alpha \left[\int_{\tau_\beta - \tau_\alpha}^\infty e^{-2KA\tau} d\tau\right]^3 + \int_0^\infty e^{-4KA\tau_\alpha} d\tau_\alpha \int_0^{\tau_\alpha} e^{+2KA\tau_\beta} d\tau_\beta$$

$$\left[\int_0^\infty e^{-2KA\tau} d\tau\right]^3. \qquad 4.27$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

The first term on the right hand side of Equation 4.27

$$= \left(\frac{1}{2KA}\right)^3 \int_0^\infty \int_0^{\tau\beta} [e^{-6KA(\tau\beta-\tau\alpha)}] e^{2KA\tau\beta} \cdot e^{-4KA\tau\alpha} d\tau_\beta d\tau_\alpha$$

$$= \left(\frac{1}{2KA}\right)^3 \int_0^\infty e^{-4KA\tau\beta} d\tau_\beta \int_0^{\tau\beta} e^{2K\tau\alpha} d\tau_\alpha$$

$$= \left(\frac{1}{2KA}\right)^3 \frac{1}{2KA} \int_0^\infty e^{-4KA\tau\beta} d\tau_\beta [e^{2KA\tau\beta} - 1]$$

$$= \left(\frac{1}{2KA}\right)^4 \left[\int_0^\infty e^{-2KA\tau\beta} d\tau_\beta - \int_0^\infty e^{-4KA\tau\beta} d\tau_\beta\right]$$

$$= \left(\frac{1}{2KA}\right)^4 \left[\frac{1}{2KA} - \frac{1}{4KA}\right]$$

$$= \frac{1}{2} \left(\frac{1}{2KA}\right)^5.$$

The second term on the right hand side of Equation 4.27

$$= \left(\frac{1}{2KA}\right)^3 \cdot \left(\frac{1}{2KA}\right) \int_0^\infty [e^{2K\tau\alpha} - 1] e^{-4K\tau\alpha} d\tau_\alpha$$

$$= \frac{1}{2} \left(\frac{1}{2KA}\right)^5.$$

Therefore expression 4.26

$$= 2 \cdot \frac{1}{2} \left(\frac{1}{2KA}\right)^5$$

$$= \left(\frac{1}{2KA}\right)^5,$$

and it follows

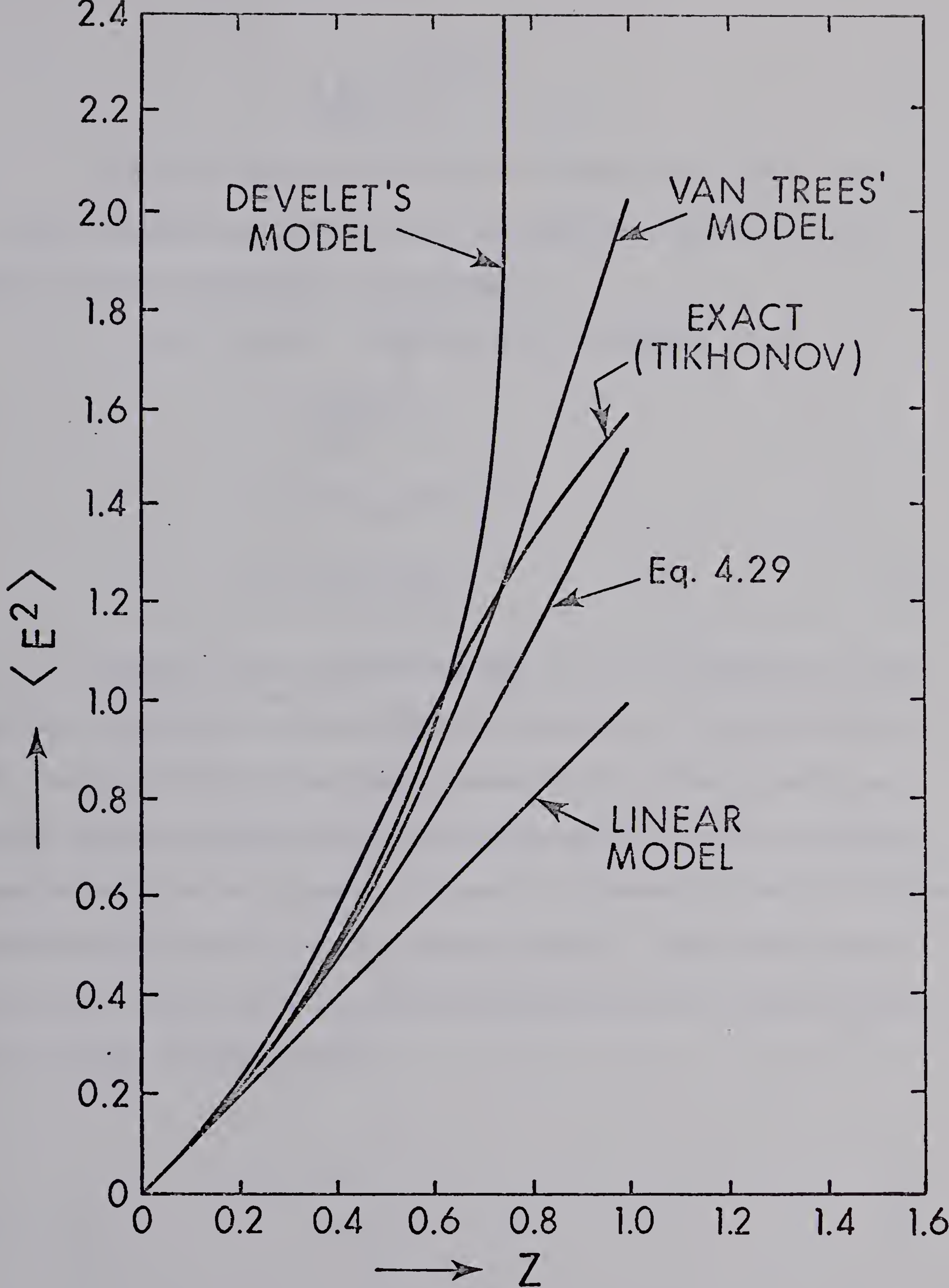
$$\langle \beta^2 E_o{}^6 \rangle = K^6 \left(\frac{N_o}{2}\right)^3 \left[\frac{9}{16K^5A^5} + \frac{6}{2^5K^5A^5}\right]$$

$$= K\left(\frac{N_o}{2}\right)^3 \cdot \frac{3}{4A^5},$$

whence

$$\left\langle \frac{K^2A^2}{3!\,3!} \beta^2 E_o{}^6 \right\rangle = \frac{K^2A^2}{36} \cdot K\left(\frac{N_o}{2}\right)^3 \frac{3}{4A^5}$$

$$= \frac{1}{6} \left(\frac{KN_o}{4A}\right)^3$$



DEVELET'S MODEL
VAN TREES' MODEL
EXACT (TIKHONOV)
Eq. 4.29
LINEAR MODEL
⟨E²⟩
Z
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4

FIGURE 4.6

$$= \frac{1}{6}Z^3. \qquad 4.28$$

To get an expression for $\langle E_1{}^2 \rangle$ in terms of the third power of Z only, the terms 4.8d and 4.8f can be neglected because they give rise to terms involving Z^4. Therefore

$$\langle E_1{}^2 \rangle = \langle E_o{}^2 \rangle + \langle 2\frac{KA}{3!}\beta(E_o)^3 E_o \rangle + \langle -2\frac{KA}{5!}(\beta E_o{}^5)(E_o) \rangle$$

$$+ \langle \frac{K^2A^2}{3!3!}\beta^2 E_o{}^6 \rangle$$

$$= Z + \frac{1}{2}Z^2 - \frac{1}{8}Z^3 + \frac{1}{6}Z^3$$

$$= Z + \frac{1}{2}Z^2 + \frac{1}{24}Z^3. \qquad 4.29$$

Equation 4.29 is plotted in Fig. 4.6 and a comparison is made between the results obtained by various methods. A noteworthy feature of Equation 4.29 which has been obtained by the Volterra functional series approach through the iteration proceedure, is the close agreement between the exact results obtained by Tikhonov[9] through the Fokker-Planck equation and the Markov process approach. The results agree even at low signal to noise ratios in which region the other methods fail to give reliable results.

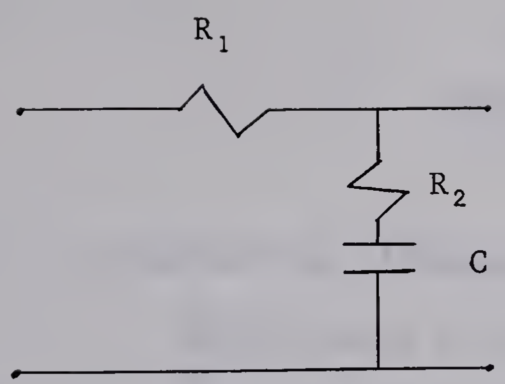
CHAPTER V

CONCLUSIONS AND REMARKS

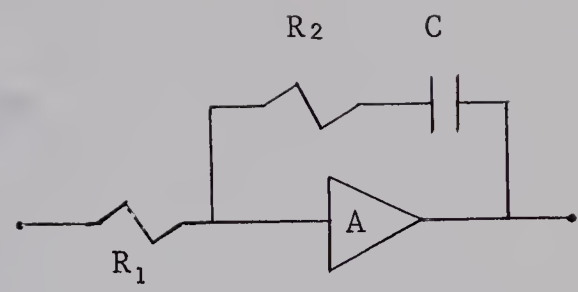
An approach invoking probabilistic functional analysis has been used for the analysis of nonlinear feedback systems with random inputs. A number of restrictions on the stochastic input (e.g. stationarity, ergodicity, normality) have been put to obtain results. The general framework of the problem is: if the input $y(\omega,t)$ is a given stochastic process and $T(\omega)$ a random operator, what type of stochastic process $\{x(\omega,t),\ t \geq 0\}$ is generated by the solution $x(\omega,t) = T^{-1}(\omega)[y(\omega,t)]$? This is a field of active mathematical research[40] whose results can be usefully applied to engineering problems.

An assumption has been made throughout that the input signal to the nonlinear element is Gaussian. But the distortion terms (which have non-Gaussian amplitude distribution) are correlated with the input signal to the system. This point has been discussed by Smith[7] where he states that in case results based on this assumption differ considerably from those obtained by quasi-linearisation, the analysis acts as a confidence test for a quasi-linear method, rather than as a method yielding higher accuracy.

The analysis of Chapter IV has indicated the usefulness of the approach in the case of the phase-locked loop systems. A first order loop without the low pass filter in the loop circuit was considered. A problem of great practical significance is the extension of the method to the case where the loop filter is also included in the phase-locked loop. Two widely used loop filters are the following. The



Passive Filter



Active Filter

Figure 5.1

passive filter is often satisfactory for many purposes whereas the active filter provides better tracking performance. Instances of the use of a filter with transfer function $\frac{(s\tau_2 + 1)(s\tau_4 + 1)}{(s\tau_1 + 1)(s\tau_2 + 1)}$ and its variants also exist in the literature[29].

APPENDIX I

VAN TREES APPROACH*

Consider the differential equation

$$\frac{dE}{dt} + K \sin E = \dot{\theta}_1(t) - KN(t)$$
$$\equiv x(t). \qquad 1$$

Suppose

$$E(t) = \sum_{i=1}^{\infty} E_i(t) \qquad 2$$

where E_1 is the output of the linear system $h_1(\tau_1)$, E_2 is the output of the second order system $h_2(\tau_1,\tau_2)$ and so on. Substitute 2 into 1 and sort the terms according to the order in which they involve x(t). Expand sin E(t) also. Therefore

$$[\dot{E}_1(t) + \dot{E}_2(t) + \dot{E}_3(t) +..] + K[\left(E_1(t) + E_2(t) + E_3(t) + \ldots\right)$$
$$- \frac{1}{3!}\left(E_1(t) + E_2(t) + E_3(t) + \ldots\right)^3 + \frac{1}{5!}\left(E_1(t) + E_2(t) + ..) - \ldots\right)^5]$$
$$= x(t). \qquad 3$$

Equate terms of equal order in x(t)

$$\dot{E}_1(t) + KE_1(t) = x(t). \qquad 4$$

All terms in 4 are first order in x(t)

$$\dot{E}_2(t) + KE_2(t) = 0 \qquad 5$$

$$\dot{E}_3(t) + KE_3(t) = \frac{K}{3!}E_1{}^3(t) \qquad 6$$

$$\dot{E}_4(t) + KE_4(t) = \frac{K}{3!}3E_1{}^2(t)E_2(t)$$
$$= \frac{K}{2}E_1{}^2(t)E_2(t). \qquad 7$$

$$\dot{E}_5(t) + KE_5(t) = \frac{K}{2}E_1{}^2(t)E_3(t) - \frac{K}{5!}E_1{}^5(t) \qquad 8$$

The equations 4 to 8 can be solved sequentially.

$$E_1(t) = \int_0^{\infty} e^{-K\tau}x(t - \tau)d\tau$$
$$= \int_0^{\infty} h_1(\tau)x(t - \tau)d\tau \qquad 9$$

---

* Reference 34.

where we define

$$h(\tau) \equiv e^{-K\tau} \qquad \tau \geq 0$$
$$= 0 \qquad \tau < 0 \qquad\qquad 10$$

Clearly

$$E_2(t) = o \qquad\qquad 11$$

because the terms on the left hand side of Equation 5 do not involve the forcing function. Consequently all higher order even terms are zero.

$$E_3(t) = \int_0^\infty h_1(\tau_1)\frac{K}{3!}E_1{}^3(t - \tau)d\tau. \qquad\qquad 12$$

Substitute 9 into 12. This gives

$$E_3(t) = \int_0^\infty d\tau \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_0^\infty d\tau_3 [\frac{K}{3!}h_1(\tau)h_1(\tau_1)h_1(\tau_2)h_1(\tau_3)$$
$$x(t - \tau - \tau_1) \cdot x(t - \tau - \tau_2)x(t - \tau - \tau_3)]. \qquad\qquad 13$$

Thus it is easy to see that E_3 depends in a third order manner on x(t).

Similarly

$$E_5(t) = \int_0^\infty h_1(\tau)[\frac{K}{2}E_1{}^2(t - \tau)E_3(t - \tau) - \frac{K}{5!}E_1{}^5(t - \tau)d\tau]. \qquad 14$$

Expressing in terms of x(t) only, we have

$$E_5(t) = \frac{K^2}{12}\int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \ldots \int_0^\infty d\tau_7 [h_1(\tau_1)h_1(\tau_2)\ldots h_1(\tau_7)$$
$$x(t - \tau_1 - \tau_2) \cdot x(t - \tau_1 - \tau_3)x(t - \tau_1 - \tau_4 - \tau_5)$$
$$\cdot x(t - \tau_1 - \tau_4 - \tau_6)x(t - \tau_1 - \tau_4 - \tau_7)$$
$$- \frac{K}{5!}\int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \ldots \int_0^\infty d\tau_6 h_1(\tau_1)\ldots h_1(\tau_6) \cdot$$
$$\cdot x(t - \tau_1 - \tau_2)x(t - \tau_1 - \tau_3)x(t - \tau_1 - \tau_4)$$
$$\cdot x(t - \tau_1 - \tau_5)x(t - \tau_1 - \tau_6). \qquad\qquad 15$$

Once again, the fifth order relationship between $E_5(t)$ and x(t) is clear. Van Trees assumes the series in Equation 2 converges without any proof. In the analysis presented in Chapter IV, we give a convergence proof.

Consider the approximate solution

$$E_a(t) \equiv E_1(t) + E_3(t) + E_5(t). \qquad 16$$

Assume

$$\dot{\theta}_1(t) = 0.$$

Let N(t) be a sample function from a white Gaussian process with correlation function

$$R_N(\tau) = \frac{N_o}{2A^2}\delta(\tau). \qquad 17$$

We want to evaluate the variance of $E_a(t)$. To evaluate $\langle E_a^2(t)\rangle$ consider the double sum

$$\langle E_a^2(t)\rangle = \sum_{i\,1,2,3} \sum_{j\,1,2,3} \langle E_iE_j\rangle \qquad 18$$

A fundamental quantity in the solution is

$$\frac{KN_o}{4A^2} \equiv Z \qquad 19$$

which physically represents noise to signal ratio. The terms of interest in Equation 18 are $\langle E_1^2(t)\rangle$, $\langle E_1(t)E_3(t)\rangle$, $\langle E_3^2(t)\rangle$ and $\langle E_1(t)E_5(t)\rangle$.

Evaluation of $\langle E_1^2(t)\rangle$. This is simply the linear approximation

$$\begin{aligned}\langle E_1^2(t)\rangle &= K^2\int_0^\infty\int_0^\infty h_1(\tau_1)h_1(\tau_2)\langle N(t-\tau_1)N(t-\tau_2)\rangle d\tau_1 d\tau_2\\ &= K^2\int_0^\infty\int_0^\infty e^{-K\tau_1-K\tau_2}\left(\frac{N_o}{2A^2}\right)\delta(\tau_1-\tau_2)d\tau_1 d\tau_2\\ &= K^2\left(\frac{N_o}{2A^2}\right)\frac{1}{2K}\\ &= \frac{KN_o}{4A^2}.\\ &= Z. \end{aligned} \qquad 19$$

<u>Van Trees Evaluation of the Other Terms</u>

(i) $\langle E_1^2(t)\rangle = Z$

(ii) $\langle E_1(t)E_3(t)\rangle = \frac{1}{4}Z^2$

(iii) $\langle E_3^2(t)\rangle = \frac{1}{6}Z^3$

(iv) $\langle E_1(t)E_5(t)\rangle = \frac{3}{16}Z^3.$

One can show that terms of $\langle E_3(t)E_5(t)\rangle$ and $\langle E_5{}^2(t)\rangle$ are fourth order in Z. Consider terms involving third order in Z only. From Equation 18 we get

$$\begin{aligned}\langle E_a{}^2(t)\rangle &= \langle E_1{}^2(t)\rangle + 2\langle E_1(t)E_3(t)\rangle + \langle E_3{}^2(t)\rangle \\ &\quad + 2\langle E_1(t)E_5(t)\rangle \\ &= Z + \frac{1}{2}Z^2 + \frac{13}{24}Z^3. \qquad 20\end{aligned}$$

Equation 20 is plotted in Fig. 4.6 to give a comparison between the results obtained in the thesis.

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